Random Walks on Hyperbolic Groups

Contents

1. Introduction

Let G be a countable group and μ be a probability measure on G. A random walk on G with step law μ is the process $W_n = G_1 \dots G_n$ where $(G_n)_{n\geq 1}$ are independent, identically distributed random elements of G with law μ. Formally, the sequence $(G_n)_{n\geq 1}$ takes values in $G^{\mathbb{N}}$ equipped with the product σ-algebra. The law of this full sequence of *increments/steps* is $\mu^{\otimes \mathbb{N}}$. We call $(G^{\mathbb{N}}, \mu^{\otimes \mathbb{N}})$ the *step space*. Consider the map $\mathbf{walk} : G^{\mathbb{N}} \to G^{\mathbb{N}}$ mapping each step sequence $(g_n)_{n\geq 1}$ to $(w_n = g_1 \dots g_n)_{n\geq 1}$. $(W_n)_{n\geq 1} = \mathbf{walk}((G_n)_{n\geq 1})$. We denote the law of a random *sample path*, by $\mathbb{P} = \mathbf{walk}_*(\mu^{\otimes \mathbb{N}})$. We will call the measurable space $\Omega = G^{\mathbb{N}}$ equipped with the probability measure $\mathbb{P}$ the *path space*. Now suppose G acts on a metric space (X, d) by isometries and let $o \in X$ be a base-point. We are interested in understanding the *asymptotic* properties of a random sample path $(W_n \cdot o)_{n\geq 1}$. Before we proceed to discuss these properties, we list some basic examples to keep in mind:

Examples:

(1) $G = \mathbb{Z}, X = \mathbb{R}, o = 0$ and $\mu = \frac{1}{2}(\delta_1 + \delta_{-1})$.
(2) $G = \mathbb{F}_2 = \langle a, b\rangle$ the free group on two generators. Let X be the Cayley graph of $\mathbb{F}_2$ with respect to the set $\{a, b\}$ and $o = e$. X is a 4-valent tree. Consider $\mu = \frac{1}{4}(\delta_a + \delta_{a^{-1}} + \delta_b + \delta_{b^{-1}})$.
(3) Take G to be any countable subgroup of $\mathrm{PSL}(2, \mathbb{R})$, the group of orientation-preserving isometries of $\mathbb{H}^2$, $X = \mathbb{H}^2$ and $o = i$. For instance, pick any $A, B \in \mathrm{PSL}(2, \mathbb{R})$ and consider $G = \langle A, B\rangle$ with $\mu = \frac{1}{2}(\delta_A + \delta_B)$.
(4) $G = \mathbb{F}_\infty$, the free group on countably infinite generators and let X be its cayley graph. X is still a tree but no longer locally-finite, it is an *infinite volume tree*. Note that X is not proper.

Remark. $(1), (2)$ are examples of a simple random walk. In all examples above, the action of G on X is the left regular action.

Perhaps the most basic question one can ask about the long-term behaviour of a random sample path is whether it almost surely travels back to its starting point, o, infinitely often or not.

Q1: Do typical sample paths escape to infinity?

Formally, we are asking if the random walk is recurrent or transient. For example, the simple random walk on $\mathbb{Z}^d$ is recurrent iff $d = 1, 2$ while the simple random walk on $\mathbb{F}_2$ is transient.

Q2: Does the random walk have positive *speed*?

Assume that the random has finite *first moment* i.e. $E[d(o, W_1 \cdot o)] = \int_G d(o, g \cdot o) d\mu(g)$ is finite, then $l = \lim_{n\to\infty} d(o, W_n \cdot o)/n$ exists $\mathbb{P}$-almost surely (cf. proposition 2.7). We have asked if $l > 0$ or not. For example, the simple random walk on $\mathbb{Z}^d$ has $l = 0$ while the simple random walk on $\mathbb{F}_2$ has $l = \frac{1}{2}$.
Now suppose the random walk is transient. If the ambient space comes with a compactification/bordification, we may ask:

Q3: Does the random walk converge to the boundary?

Where recall that a *bordification* $\bar{X}$ of X is a Hausdorff, second-countable topological space together with an embedding $X \hookrightarrow \bar{X}$ such that the G-action on X extends to $\bar{X}$ by homeomorphisms. A bordification will be called a *compactification* if $\bar{X}$ is compact. The *boundary* of X is $\partial X = \bar{X} - X$. Now here's an answer to **Q3**:

Theorem ([MT18])**.** *Let μ be a non-elementary measure on a countable group G which acts by isometries on a δ-hyperbolic metric space (X, d). Then for all $o \in X$, for $\mathbb{P}$-almost every sample path $(w_n \cdot o)$ the limit $\lim_{n\to\infty} w_n \cdot o \in \partial X$ exists.*

When G is hyperbolic and X is its Cayley graph wrt to a finite generating set, the above theorem was proved in [Kai94]. We will see this proof in proposition 6.4.
When X comes with a bordification $\bar{X}$ and the random walk converges to the boundary, we can try to understand the subset of the boundary it converges to by means of the *hitting measure* defined as $\nu_\mu(A) = \mathbb{P}(\lim_{n\to\infty} W_n \cdot o \in A)$ for all Borel subsets A of ∂X. It can be easily seen that μ_ν is μ-stationary. What are the properties of the hitting measure?

Q4: What is the Hausdorff dimension of the hitting measure?

We discuss this question and compute the Hausdorff dimension in a special case following [BHM11] in section 5. When $G < \mathrm{SL}(2, \mathbb{R})$ and $X = \mathbb{H}^2$, we have have $\partial X \simeq S^1$ and thus we can ask:

Q5: Is ν_μ in the Lebesgue measure class?

In relation to **Q5**, we mention the following conjecture adapted from [KLP11].

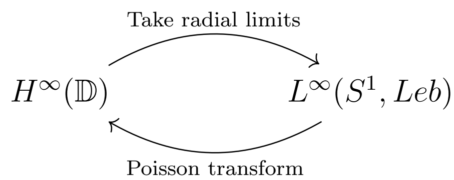
Conjecture (Singularity conjecture)**.** *For any finitely supported probability measure μ on $SL(2, \mathbb{R})$ such that the group generated by the support of μ is discrete, the measure ν_μ is singular wrt to the Lebesgue measure.*

We will now introduce the main problem that this note is concerned about: the Poisson boundary indentification problem. Let us reset our minds to back when we were considering the basic question of recurrence/transience of a random walk, before **Q3**. Suppose our random walk is transient and furthermore, the measure μ is non-degenerate i.e. the support of μ generates G as a semi-group. Since the random walk is transient, $\mathbb{P}$-almost every sample path escapes to infinity. From a geometric viewpoint, we expect the sample path to travel to a *boundary at infinity*. Inspired by this, it is possible to come up with a purely measure-theoretic notion of boundary. Consider a measurable space $(B, \mathcal{F}_B)$ together with a measurable map $\mathbf{bnd} : \Omega \to B$. Given any sample path $\mathbf{w} = (w_n)$ we would like to think of $\mathbf{bnd}(\mathbf{w})$ as representing the *point at infinity* it escapes to. To be able to call $(B, \mathcal{F}_B, \mathbf{bnd})$ a *candidate boundary*, we require it satisfy the following conditions:

- $\mathbf{bnd}$ is invariant under the *time-shift* map $T : \Omega \to \Omega$, $T((w_n)) = (w_{n+1})$, i.e. $\mathbf{bnd} \circ T = \mathbf{bnd}$. We require this because if (w_n) escapes to a point $b \in B$ then it still does so if we forget its location at some finitely many times.
- $\mathcal{F}_B$ is countably generated and *separates points* of B. This is analogous to bordifications being second countable.
- Since G acts on Ω component-wise, let us also require $\mathbf{bnd}$ to be G-equivariant.

It is therefore natural to ask: is there a maximal candidate boundary for the random walk? This is the initial object in the category of candidate boundaries $(B, \mathcal{F}_B, \mathbf{bnd})$ for (G, μ), i.e. it is a candidate boundary $(B_{\max}, \mathcal{F}_{B_{\max}}, \mathbf{bnd}_{\max})$ such that for any other candidate boundary $(B, \mathcal{F}_B, \mathbf{bnd})$, there exists a measurable map $f : B \to B'$ and $\mathbf{bnd} = f \circ \mathbf{bnd}_{\max}$ $\mathbb{P}$-almost surely. It turns out that the existence of such a maximal candidate boundary is related to μ-harmonic functions on G. Henceforth, a candidate boundary will be called a μ-boundary for (G, μ).

Recall the Poisson representation formula from complex analysis. Let $H^\infty(\mathbb{D})$ denote the set of bounded harmonic functions on $\mathbb{D}$, the unit disc in $\mathbb{C}$. Then we have a *duality*, an isometric linear isomorphism between $H^\infty(\mathbb{D})$ and $L^\infty(S^1, Leb)$.



Where the Poisson transform is given as follows: If $f \in L^\infty(S^1, Leb)$ then $u : \mathbb{D} \to \mathbb{R}$ defined by

$$\boxed{u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})P(r, t-\theta)dt} \quad (*)$$

is harmonic. Here $P(r,t)$ is the Poisson kernel:

$$P(r,t) = \frac{1-r^2}{1+r^2-2r\cos(t)}$$

We will now re-write $(*)$. Let $a = re^{i\theta}$ and g be a conformal automorphism of $\mathbb{D}$ which maps 0 to a. This is a Blaschke map, for eg, consider $g(z) = \frac{a-z}{1-\bar{a}z}$. Putting $z = e^{it}$, we see that:

$$|g'(z)| = \frac{1-|a|^2}{|1-\bar{a}z|^2} = \frac{1-r^2}{|1-re^{i(t-\theta)}|^2} = P(r, t-\theta)$$

Thus the Poisson representation formula $(*)$ can be re-written as follows:

$$u(re^{i\theta}) = \int_0^{2\pi} f(e^{it})|g'(e^{it})|\frac{dt}{2\pi}$$

Since $dt/2\pi$ is the Lebesgue measure λ on $\partial D = S^1$, we have:

$$u(a) = \int_{\partial\mathbb{D}} f(\xi)dg_*\lambda(\xi)$$

We observe that the ingredients of the Poisson representation formula are: boundary data (f on $\partial\mathbb{D}$), the group Aut($\mathbb{D}$) and an Aut($\partial\mathbb{D}$)-invariant measure on $\mathbb{D}$ (the Lebesgue measure on $\partial\mathbb{D}$). We now transition from Aut($\mathbb{D}$) to the measured group (G,μ) as before. A function $f : G \to \mathbb{R}$ is called μ-harmonic if for all $g \in G$, $f(g) = \sum_{h\in G}\mu(h)f(gh)$. Let $H^\infty(G,\mu)$ denote the Banach space of bounded μ-harmonic functions on G. It is natural to therefore try to understand the relationship between the geometry/algebra of groups and the space $H^\infty(G,\mu)$. Let us list some examples:

Examples:

(1) $G = \mathbb{Z}, \mu = \frac{1}{2}(\delta_1 + \delta_{-1})$ (simple random walk). Then:

$$H(G,\mu) = \{\text{affine functions}\} = \{f : \mathbb{Z} \to \mathbb{R} \mid \forall n, f(n) = an + b, a, b \in \mathbb{R}\}$$

Thus the only bounded harmonic functions on Z are constants, $H^\infty(G,\mu) = \mathbb{R}$.

(2) $G = \mathbb{Z}, \mu = \frac{1}{q+1}\delta_{-1} + \frac{q}{q+1}\delta_1$ (biased random walk) where $q > 1$ is some integer. Then:

$$H(G,\mu) = \{f : \mathbb{Z} \to \mathbb{R} \mid \forall n, f(n) = aq^{-n} + b, a, b \in \mathbb{R}\}$$

Thus again, the only bounded harmonic functions on Z for the biased random walk are constants.

(3) $G = \mathbb{F}_2 = \langle a, b\rangle$. Let X be the Cayley graph of G wrt the standard generating set and consider the simple random walk. Let $C(a)$ denote the words in G starting with a. Then

$$f(g) = \begin{cases} 3^{-|g|+1} & g \in C(a) \\ -3^{-|g|} + 4 & g \notin C(a) \end{cases}$$

is a non-constant bounded harmonic function on G.

Given any μ-boundary $(B, \mathcal{F}_B, \mathbf{bnd})$ for (G, μ), we define $\nu = \mathbf{bnd}_*\mathbb{P}$. In analogy with the Poisson representation formula for harmonic functions on the unit disk, we define the *Poisson transform* $\Phi : L^\infty(B, \nu) \to H^\infty(G, \mu)$:

$$\forall f \in L^\infty(B, \nu), \forall g \in G, \Phi(f)(g) := \int_B f dg_*\nu$$

Given (G, μ), a μ-boundary $(B, \mathcal{F}_B, \mathbf{bnd})$ is called the *Poisson boundary* of (G, μ) if the Poisson transform is bijective. In fact, we show in theorem 6.1, $(B, \mathcal{F}_B, \mathbf{bnd})$ is maximal if and only if the Poisson transform is bijective.

Remark. Thus the Poisson boundary of (G, μ) is trivial if and only if every bounded μ-harmonic function on G is constant.

Given a group G and a non-degenerate probability measure μ on G, we want to identify its Poisson boundary. In this broad sense, it is extremely challenging. We ask some more precise questions:

Q6: Are there groups G such that (G, μ) has trivial Poisson boundary for all $\mu \in \text{Prob}(G)$?

Such groups are called *Choquet-Deny* groups and have been characterized completely in [FHTVF19]. On the other hand, G is non-amenable if and only if the Poisson boundary of (G, μ) is non-trivial for any non-degenerate $\mu \in \text{Prob}(G)$ (cf. [KV83]).

Q7: When G admits a compactification/bordification and μ is non-degenerate and non-elementary, is ∂G equipped with the hitting measure a model for the Poisson boundary of (G, μ)?

Kaimanovich's entropy criterion (cf. [KV83],[Kai00], theorem 7.3) has proved to be a simple, robust and important tool/guiding principle in solving **Q7**. Using the strip approximation technique in [Kai00], he proves:

Theorem. *Let G be a non-elementary hyperbolic group. Then for any $\mu \in Prob(G)$ which is non-degenerate, with (1) finite entropy and (2) finite logarithmic moment, the Poisson boundary of (G, μ) is the Gromov boundary of G.*

(1) $H(\mu) = -\sum_{g\in G} \mu(g) \log(\mu(g)) < \infty$.
(2) $\sum_{g\in G} \log^+ d(o, g \cdot o)\mu(g) < \infty$.

We give a proof of this theorem in section 7 using theorem 9.2 and theorem 9.3. The strip approximation technique crucially needs the finiteness of the first logarithmic moment of the random walk. Recently, [CFFT22] has removed the logarthmic moment condition:

Theorem. *For any finite entropy, non-degenerate probability measure μ on a non-elementary hyperbolic group G, the Gromov boundary is the Poisson boundary of (G, μ).*

Their argument is a wonderful application of theory of pivots introduced by Gouezel in [Gou22] and transformed random walks (cf. [For15],[For17]). We systematically review the theory of pivots as required for their argument in section 10. In section 11, we briefly talk about transformed random walks, a stronger entropy criterion and the pin-down approximation method of [CFFT22] to prove the theorem.

Reference chart:

- **Section 1** Heavily influenced by the set of lectures delivered by Giulio Tiozzo at Probabilistic Methods in Negative Curvature, ICTS Bangalore, 2023, [Tio].
- **Section 2** This section is closely based on Peter Haïssinski's lectures at Probabilistic Methods in Negative Curvature, ICTS Bangalore, 2019, [Haï].
- **Section 3, 4, 5** [Haï] and [BHM11].
- **Section 6** This section is influenced by Vadim A. Kaimanovich's lectures at Probabilistic Methods in Negative Curvature, ICTS Bangalore, 2019, [Kai].
  - 6.1: [Kai00], 6.2: [LP16], 6.3, 6.4: [LP16] and [Woe00].
  - 6.5: [Mar91], 6.6: [Kai00]
- **Section 7** [Kai00], [KV83], [LP16].
- **Section 8, 9** [Kai00].
- **Section 10** [Gou22].
- **Section 11**
  - 11.1: [For17] and [For15].
  - 11.2, 11.3: [CFFT22] and [Tio].

Acknowledments. I would like to express my gratitude to Prof. Mahan Mj. for his patience and guidance. I am also grateful to my friend Balarka Sen for listening to my thoughts on entropy and random walks. I thank both of them for many insightful discussions. I also acknowledge that my interest in the topic was greatly energized by interactions with Prof. Giulio Tiozzo at the Probabilistic Methods in Negative Curvature workshop held at ICTS, Bangalore in March, 2023.

2. Random Walks on Countable Discrete Groups

Let Γ be a topological space. How to "walk" on Γ? Irrespective of how one walks on Γ, clearly it produces for us a sequence $(x_n)_{n\geq 0}$ of elements in Γ where x_n is the location of our walk after the n^{th} step has been made. Thus a random walk on Γ could be prescribed by a random variable say Z taking values in $\Gamma^{\mathbb{N}}$ the $\mathbb{N}$-fold product of Γ, equipped with the product topology, the space of *sample* paths in Γ. Keeping this in mind, we specialize to the case when Γ is a countable group equipped with the discrete topology. Here we can utilize the group structure of Γ to walk on it. More precisely:

- Let μ be a Borel probability measure on Γ. Since Γ is discrete, μ is entirely determined by its values on point sets. Thus, we have a function $\mu : \Gamma \to \mathbb{R}_{\geq 0}$ such that $\sum_{g\in\Gamma} \mu(g) = 1$.
- Let Ω denote the product space $\Gamma^{\mathbb{N}_{>0}}$, equipped with the Borel σ-algebra $\mathcal{B}$ and the product probability measure $P = \mu^{\otimes \mathbb{N}_{>0}}$. We will think of Ω as a space of *increments*. A typical element of Ω will be denoted $\omega = (\omega_n)_{n\geq 1}$.
- For each $n \in \mathbb{N}_{>0}$, let $X_n : \Omega \to \Gamma$ be the projection onto n^{th} coordinate, i.e. $X_n((\omega_m)_{m\geq 1}) = \omega_n$ for all $(\omega_m)_{m\geq 1} \in \Omega$. Observe that $(X_n)_{n\geq 1}$ is a sequence of Γ-valued i.i.d. random variables with law μ.

- Then the random walk on Γ with law μ is given by a sequence $(Z_n)_{n\geq 0}$ of Γ-valued random variables defined by: $Z_0 \equiv e$, $Z_{n+1} = Z_n X_{n+1}$ for $n \in \mathbb{N}$, where e is the identity element of Γ.

Definition 2.1. The **random walk on** Γ with **law** μ, denoted by $\mathrm{RW}_{\Gamma,\mu}$ refers to the following data: $(\Omega, \mathcal{B}, P, \mu, (X_n)_{n\geq 1})$, together with the sequence $(Z_n)_{n\geq 0}$ defined in terms of $(X_n)_{n\geq 1}$ as above.

Recall that:

Definition 2.2. Suppose $(M, \mathcal{A}, \nu), (N, \mathcal{B}, \eta)$ are probability spaces and $(Q, \mathcal{C})$ is a measurable space. Suppose $A : M \times N \to Q$ is a measurable function, where $M \times N$ has been equipped with the product σ-algebra $\mathcal{A} \otimes \mathcal{B}$ and product probability measure $\nu \otimes \eta$. The **convolution** of ν and η, denoted by $\nu * \eta$ is the push-forward of $\nu \otimes \eta$ under A. Thus for any bounded measurable function $\varphi : Q \to \mathbb{R}$:

$$\begin{aligned}\int_Q \varphi d(\nu * \eta) &= \int_{M\times N} \varphi \circ A d(\nu \otimes \eta) \\ &= \int_N \int_M \varphi(A(x, y)) d\nu(x) d\eta(y)\end{aligned}$$

Remark 2.1. The sequence $(Z_n)_{n\geq 0}$ is to be thought of as a random sample path in Γ starting at the identity element. For $n \in \mathbb{N}$, Z_n is the location of our walk after the n^{th} step has been made. At each step, an element of Γ is sampled according to the law μ and multiplied with our current location on the right so as to move to a possibly new group element. A random walk starting at $g \in \Gamma$ is simply given by the sequence $(gZ_n)_{n\geq 0}$. Let us observe that:

- The law of Z_n is $\mu^n := \mu^{*n}$, the n-fold convolution of μ.
- Let $\mathcal{T} = \Gamma^{\mathbb{N}}$ equipped with the Borel σ-algebra. We think of $\mathcal{T}$ as the space of *sample paths* in Γ as follows: given a sequence of increments $\omega \in \Omega$ and a starting point $g \in \Gamma$, we get the sample path $(gZ_n(\omega))_{n\geq 0} \in \mathcal{T}$. Thus $\mathrm{RW}_{\Gamma,\mu}$ allows us to sample paths from $\mathcal{T}$ according to the law $\mu * P$.

The following proposition highlights the associativity of the convolution operation and in particular the Markov property of $\mathrm{RW}_{\Gamma,\mu}$.

Proposition 2.1. *For all $m, n \in \mathbb{N}_{>0}$ and for all $x \in \Gamma$, $\mu^{m+n}(x) = \sum_{y\in\Gamma} \mu^m(y)\mu^n(y^{-1}x)$.*

Proof.

$$\begin{aligned}
\mu^{m+n}(x) &= P[Z_{m+n} = x] \\
&= \sum_{y\in\Gamma} P[Z_{m+n} = x, Z_m = y] \quad \text{(law of total probability)} \\
&= \sum_{y\in\Gamma} P[Z_m^{-1} Z_{m+n} = y^{-1}x, Z_m = y] \\
&= \sum_{y\in\Gamma} P[\prod_{i=1}^{n} X_{m+i} = y^{-1}x, Z_m = y] \\
&= \sum_{y\in\Gamma} P[Z_m = y]P[Z_n = y^{-1}x] \quad (Z_m \text{ and } \prod_{i=1}^{n} X_{m+i} \text{ are independent}, \prod_{i=1}^{n} X_{m+i} \overset{d}{=} Z_n) \\
&= \sum_{y\in\Gamma} \mu^m(y)\mu^n(y^{-1}x)
\end{aligned}$$

■

Definition 2.3 (Hitting times)**.** Let $A \subset \Gamma$ and $x, y \in \Gamma$. We define the following **hitting times**:

$$\begin{aligned}
\tau^0_{x,A} &= 0 \\
\tau^k_{x,A} &= \min\{n > \tau^{k-1}_{x,A} \mid xZ_n \in A\} \text{ for } k \geq 1
\end{aligned}$$

We will write τ^k_{xy} when $A = \{y\}$.

We are interested in studying asymptotic properties of random sample paths. Perhaps the simplest questions to be asked are those of recurrence.

Definition 2.4 (Recurrence, Transience, Irreducibility, Symmetry)**.** $\mathrm{RW}_{\Gamma,\mu}$ is said to be:

- **recurrent** if almost surely, a sample path starting at e returns to e infinitely often, i.e. $P[Z_n = e \text{ i.o.}] = 1$ or equivalently, if $P[\tau^1_{ee} < \infty] = 1$. Otherwise, $\mathrm{RW}_{\Gamma,\mu}$ is said to be **transient**.
- **irreducible** if the semi-group generated by $\mathrm{supp}(\mu)$ equals whole of Γ. Here $\mathrm{supp}(\mu) = \{g \in \Gamma \mid \mu(g) > 0\}$.
- **symmetric** if for all $g \in \Gamma$, we have $\mu(g) = \mu(g^{-1})$.

Definition 2.5. We define $G : \Gamma \times \Gamma \to [0, \infty]$ by $G(x, y) = \sum_{n=0}^{\infty} P[xZ_n = y]$, called the **Green's function**. $G(x, y)$ is the expected number of times a random walk starting at x visits y.

Proposition 2.2. *Let $x, y \in \Gamma$.*

(a) *If $RW_{\Gamma,\mu}$ is symmetric, then for all $n \in \mathbb{N}_{>0}$, $\mu^{2n}(x) \leq \mu^{2n}(e)$, $\mu^{2n+1}(x) \leq \mu^{2n}(e)$.*
(b) *If $RW_{\Gamma,\mu}$ is irreducible and transient then $G(x, y)$ is finite.*
(c) *If $RW_{\Gamma,\mu}$ is symmetric then $G(x, y) = G(y, x)$.*
(d) *If $RW_{\Gamma,\mu}$ is irreducible then $G(x, y) > 0$.*

Proof. (a):

$$\begin{aligned}
\mu^{2n}(x) &= \sum_{y\in\Gamma} \mu^n(y)\mu^n(y^{-1}x) \quad \text{(Proposition 1.1)}\\
&\leq \left(\sum_{y\in\Gamma}(\mu^n(y))^2\right)^{1/2}\left(\sum_{y\in\Gamma}(\mu^n(y^{-1}x))^2\right)^{1/2} \quad \text{(Cauchy-Schwarz inequality)}\\
&= \sum_{y\in\Gamma}(\mu^n(y))^2\\
&= \sum_{y\in\Gamma} \mu^n(y)\mu^n(y^{-1}) \quad (\mu \text{ is symmetric})\\
&= \mu^{2n}(e) \quad \text{(Proposition 1.1)}\\
\mu^{2n+1}(x) &= \sum_{y\in\Gamma} \mu^{2n}(y)\mu(y^{-1}x)\\
&\leq \mu^{2n}(e)\sum_{y\in\Gamma}\mu(y^{-1}x) = \mu^{2n}(e)
\end{aligned}$$

(b): Consider the **hitting probabilities** $\rho_{xy} = P_x[\tau^1_{xy} < \infty]$ where P_x is the law of the random walk starting at x. Recall from [Dur10] that the strong markov property of the random walk gives us:

$$\boxed{P_x[\tau^k_{xy} < \infty] = \rho_{xy}\rho^{k-1}_{yy}}$$

Now we define a the random variable $N(x,y) = \sum_{n=0}^{\infty} \mathbf{1}_{[xZ_n=y]}$ which counts the number of times a random walk starting at x visits the point y. Note that $G(x,y) = E[N(x,y)]$. Now we observe that:

$$\begin{aligned}
E[N(x,y)] &= \sum_{n=0}^{\infty} P[N(x,y) \geq k]\\
&= \sum_{n=0}^{\infty} P[\tau^k_{xy} < \infty]\\
&= \sum_{n=0}^{\infty} \rho_{xy}\rho^{n-1}_{yy}\\
&= \frac{\rho_{xy}}{1-\rho_{yy}} = \frac{\rho_{xy}}{1-\rho_{ee}}
\end{aligned}$$

which is finite if the random walk is irreducible (implies ρ_{xy} is finite) and transient.
(c): Observe that μ is symmetric implies that μ^n is symmetric for all $n \geq 1$. Then the claim immediately follows.
(d): Since the random walk is irreducible, the element $x^{-1}y$ belongs to the semigroup generated by the support of μ, which means that there is a $n \geq 1$ for which $\mu^n(x^{-1}y) > 0$, so $G(x,y) > 0$. ∎

Claim 2.1. Let $x \in \Gamma$. Then $\limsup_{n\to\infty}(\mu^{2n}(x))^{1/2n} = \limsup_{n\to\infty}(\mu^{2n}(e))^{1/2n}$

Proof. ∎

Thus it makes sense to define:

Definition 2.6. For a random walk $\mathrm{RW}_{\Gamma,\mu}$, we define its **spectral radius** to be $\rho(\mu) = \limsup_{n\to\infty}(\mu^{2n}(e))^{1/2n}$.

Theorem 2.1 (Kesten's Criteria)**.** *Let Γ be a finitely generated group and μ be a Borel probability measure on Γ. Then:*

(a) *If $RW_{\Gamma,\mu}$ is irreducible and $\rho(\mu) = 1$ then Γ is amenable.*
(b) *If $RW_{\Gamma,\mu}$ is irreducible, symmetric and Γ is amenable then $\rho(\mu) = 1$.*

For a proof, refer to [Woe00] section 12.

Corollary 2.1.1. *Let Γ be a finitely generated, non-amenable group and μ be a Borel probability measure on Γ such that $RW_{\Gamma,\mu}$ is irreducible. Then there exist constants $C, m > 0$ such that:*

$$\forall x \in \Gamma, \forall n \geq 1, \mu^{2n}(x) \leq Ce^{-mn}$$

Definition 2.7. We define $F : \Gamma \times \Gamma \to [0, 1]$ by $F(x, y) = P[\exists n \in \mathbb{N}, xZ_n = y]$, the **hitting probability** that a random walk starting at x visits y.

Proposition 2.3. *Let $x, y, z \in \Gamma$. Then:*

(a) $G(x, y) = F(x, y)G(e, e)$
(b) $F(x, y) \geq F(x, z)F(z, y)$

Proof. We have already proved (a) in proposition 2.2 (b) using the boxed relation preceding the proposition. Let us give a direct proof:

$$\begin{aligned} G(x,y) &= \sum_{n=0}^{\infty} P[xZ_n = y] \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} P[xZ_n = y, \tau^1_{xy} = k] \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} P[xZ_k = y, \tau^1_{xy} = k, X_{k+1}\dots X_n = e] \end{aligned}$$

Now observe that the events $[xZ_k = y, \tau^1_{xy} = k]$ and $[X_{k+1} \dots X_n = e]$ are independent. Also, that the random variables $X_{k+1} \dots X_n$ and Z_{n-k} are distributionally equivalent. Thus:

$$
\begin{aligned}
G(x,y) &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} P[xZ_k = y, \tau^1_{xy} = k, X_{k+1} \dots X_n = e] \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} P[xZ_k = y, \tau^1_{xy} = k] P[Z_{n-k} = e] \\
&= \sum_{k=0}^{\infty} \sum_{n \geq k} P[xZ_k = y, \tau^1_{xy} = k] P[Z_{n-k} = e] \\
&= \sum_{k=0}^{\infty} P[xZ_k = y, \tau^1_{xy} = k] \left(\sum_{n \geq k} P[Z_{n-k} = e] \right) \\
&= \sum_{k=0}^{\infty} P[xZ_k = y, \tau^1_{xy} = k] G(e,e) \\
&= F(x,y) G(e,e)
\end{aligned}
$$

(b) is immediate. ■

Proposition 2.4. *Suppose $RW_{\Gamma,\mu}$ is transient, symmetric and irreducible. Then $d_G(x,y) := -\log(F(x,y))$ for all $x, y \in \Gamma$, defines a metric on Γ. Furthermore:*

(a) *(Γ, d_G) is a proper metric space.*
(b) *Γ acts on (Γ, d_G) from the left, by isometries.*

Proof.

- Since μ is symmetric, so is $F(-,-)$ and thus d_G is symmetric. Alternatively, it follows from proposition 2.2(c) and proposition 2.3(a).
- Taking negative logarithm on both sides of the inequality in proposition 2.3(b) gives the triangle inequality for d_G.
- Proposition 2.2(d) implies that d_G is finite. Transience of the random walk ensures that $d_G(x,y) = 0$ implies $x = y$. After all $d_G(x,y) = 1$ if and only if $F(x,y) = 1$. So if $x \neq y$, we have $1 \geq \rho_{xx} \geq F(x,y)F(y,x) = 1$, so $\rho_{xy} = 1$ which contradicts transience.
- Thus d_G is a metric.
- Clearly F is Γ-invariant under the diagonal action. Thus, d_G is Γ-invariant under the diagonal action. It remains to show that d_G is proper. Observe that it is enough to show:

$$
\boxed{\forall \varepsilon \in (0,1), \exists K \subset_{\text{finite}} \Gamma, \text{ such that } \forall x \notin K, G(e,x) \leq \varepsilon}
$$

- Fix $\varepsilon \in (0,1)$. Since the random walk is transient, $G(e,e)$ is finite. $G(e,e) = \sum_{n=1}^{\infty} \mu^{2n}(e)$ is finite so there is a k large enough so that:

$$
\sum_{n \geq k} \mu^{2n}(e) \leq \varepsilon/4
$$

Now using proposition 2.2(a), we have:

$$\sum_{n\geq k}\mu^{2n}(x)\leq\sum_{n\geq k}\mu^{n}(e)\leq\varepsilon/4$$
$$\sum_{n\geq 2k}\mu^{n}(x)\leq 2\sum_{n\geq k}\mu^{n}(e)\leq\varepsilon/2$$

Because $\mu,\dots,\mu^{2k-1}$ are probability measures, there is a finite subset K such that $\mu^i(\Gamma-K)\leq\varepsilon/(4k)$ for $1\leq i<2k$. Thus for any $x\notin K$, we have:

$$\begin{aligned}G(e,x)&=\sum_{1\leq n<2k}\mu^n(x)+\sum_{n\geq 2k}\mu^n(x)\\&\leq\sum_{1\leq n<2k}\mu^n(\Gamma-K)+\frac{\varepsilon}{2}\leq\frac{\varepsilon}{2k}+\frac{\varepsilon}{2k}=\varepsilon\end{aligned}$$

■

We call d_G the **Green metric**, first introduced in [BB07].Let us now look at some dynamical aspects of $\mathrm{RW}_{\Gamma,\mu}$. Let $\sigma:\Omega\to\Omega$ be the left-shift map defined by $\sigma((\omega_n)_{n\geq 1})=(\omega_{n+1})_{n\geq 1}$.

Proposition 2.5. *P is a σ-invariant and σ-ergodic measure on Ω.*

Proof. This is pretty standard. ■

We recall the statement of the subadditive ergodic theorem:

Theorem 2.2. *Let (X,μ) be a measure space and $T:X\to X$ be a measurable map such that μ is T-invariant. Let $(f_n:X\to\mathbb{R})_{n>0}$ be a sequence of measurable functions such that:*

- $\forall m,n\in\mathbb{N}_{>0}\forall x\in X,\ f_{m+n}(x)\leq f_m(x)+f_n(T^m(x))$
- $f_1\in L^1(\mu)$.

Then, $\lim_{n\to\infty} f_n/n$ exists a.e. and denoting the limiting function by f, we also have that $f_n/n\to f$ in $L^1(\mu)$. If μ is T-ergodic, then $f=inf_{n\geq 1}E[f_n]/n$.

For a proof, see [Dur10].

Proposition 2.6 (Asymptotic entropy)**.** *Consider a random walk $RW_{\Gamma,\mu}$. If $H(\mu)=\sum_{g\in\Gamma}-\mu(g)log(\mu(g))<\infty$ then the sequence $\frac{-1}{n}log(\mu^n\circ Z_n)$ converges a.s. and in $L^1(P)$ to a constant h, which we call the* ***asymptotic entropy*** *of $RW_{\Gamma,\mu}$.*

Proof. For each $n\in\mathbb{N}_{>0}$, define $f_n=-\log(\mu^n\circ Z_n)$. Note that each f_n is a non-negative measurable function and that $\int_\Omega f_1 dP=E[-\log\mu]$ is finite. P is σ-invariant and ergodic. From proposition (1.1), it follows that for any $\omega\in\Omega$ and $m,n\geq 1$ we have:

$$\begin{aligned}\mu^{m+n}(Z_{m+n}(\omega))&\geq\mu^m(Z_m(\omega))\mu^n(Z_m(\omega)^{-1}Z_{m+n}(\omega))\\&=\mu^m(Z_m(\omega))\mu^n(\prod_{i=1}^{n}X_{m+i}(\omega))\\&=\mu^m(Z_m(\omega))\mu^n(Z_n(\sigma^m(\omega)))\end{aligned}$$

so by the subadditive ergodic theorem, the sequence $\frac{-1}{n}\log(\mu^n\circ Z_n)$ converges a.s. and in $L^1(P)$ to a constant h and $h=\inf_n\frac{-\log(\mu^n\circ Z_n)}{n}=\inf_n\frac{H(\mu^n)}{n}$. ■

Proposition 2.7 (Drift)**.** *Consider a random walk $RW_{\Gamma,\mu}$. Suppose (X,d) be a metric space and Γ acts on (X,d) by isometries. Pick any $x_0 \in X$. If $E[d(x_0, X_1 \cdot x_0)] = \sum_{g\in\Gamma} \mu(g) d(x_0, g \cdot x_0) < \infty$, then the sequence $d(x_0, Z_n \cdot x_0)/n$ converges a.s. and in $L^1(P)$ to a constant l, which we call the **drift** of the random walk.*

Proof. For each $n \in \mathbb{N}_{>0}$, define $f_n = d(x_0, Z_n \cdot x_0)$. Note that each f_n is a non-negative measurable function and that $\int_\Omega f_1 dP = E[d(x_0, X_1 \cdot x_0)]$ is finite. P is σ-invariant and ergodic. For any $\omega \in \Omega$ and $m, n \geq 1$ we have:

$$\begin{aligned} d(x_0, Z_{m+n}(\omega)\cdot x_0) &\leq d(x_0, Z_m(\omega)\cdot x_0) + d(Z_m(\omega)\cdot x_0, Z_{m+n}(\omega)\cdot x_0) \\ &= d(x_0, Z_m(\omega)\cdot x_0) + d(x_0, Z_m(\omega)^{-1} Z_{m+n}(\omega)\cdot x_0) \\ &= d(x_0, Z_m(\omega)\cdot x_0) + d(x_0, Z_n(\sigma^m(\omega))\cdot x_0) \end{aligned}$$

So by the subadditive ergodic theorem, the sequence $d(x_0, Z_n \cdot x_0)/n$ converges a.s. and in $L^1(P)$ to a constant l and $l = \inf_n \frac{E[d(x_0, Z_n \cdot x_0)]}{n}$. ■

Remark. We will be using the Borel-Cantelli lemmas quite often. [Yeo] provides a very neat review which the reader might want to take a look at.

Lemma 2.1 (Zoning lemma)**.** *Consider a random walk $RW_{\Gamma,\mu}$. Suppose $H(\mu) < \infty$. For every $\varepsilon > 0$,*

(a) There exists a sequence (A_n) of subsets of Γ such that $|A_n| \leq e^{n(h+\varepsilon)}$ and $Z_n \in A_n$ eventually a.s.
(b) If (B_n) is a sequence of subsets of Γ such that $|B_n| \leq e^{n(h-\varepsilon)}$, then $Z_n \notin B_n$ i.o. a.s.

This means that the asymptotic entropy governs the complexity of the random walk, at least in terms on the size of regions it can visit.

Proof. Let us recall that if (Ω_n) is a sequence of events in Ω then:

- $\{\Omega_n \text{ eventually }\} = \{\omega \in \Omega \mid \exists n_0 = n_0(\omega), \text{ such that for } n \geq n_0, \omega \in \Omega_n\} = \cup_n \cap_{k\geq n} \Omega_k$.
- $\{\Omega_n \text{ i.o. }\} = \{\omega \in \Omega \mid \exists (n_k) = (n_k(\omega))_k, \text{ such that for all } k, \omega \in \Omega_{n_k}\} = \cap_n \cup_{k\geq n} \Omega_k$.
- $\cap_n \Omega_n \subseteq \{\Omega_n \text{ eventually }\} \subseteq \{\Omega_n \text{ i.o. }\} \subseteq \cup_n \Omega_n$

Now note that since $H(\mu)$ is finite, proposition 2.6 guarantees the existence of a finite asymptotic entropy h, and:

$$\text{for a.e. } \omega \in \Omega, \lim_{n\to\infty} \frac{-1}{n} \log(\mu^n(Z_n(\omega))) = h \quad ...(*)$$

- Proof of (a): Let Ω' be the subset of full measure for which the limit in $(*)$ holds and consider the sets $\Omega_n = \{\omega \in \Omega \mid \mu^n(Z_n(\omega)) \geq e^{-n(h+\varepsilon)}\} \subseteq \Omega$. Then $(*)$ implies that $\Omega' \subseteq \{\Omega_n \text{ eventually }\}$, so $P(\{\Omega_n \text{ eventually }\}) = 1$. Now consider the sets $A_n = \{g \in \Gamma \mid \mu^n(g) \geq e^{-n(h+\varepsilon)}\}$. Clearly, $\Omega_n = [Z_n \in A_n]$ so $Z_n \in A_n$ eventually a.s. Also, we have $1 \geq \mu^n(A_n) = \sum_{g\in A_n} \mu^n(g) \geq |A_n| e^{-n(h+\varepsilon)}$, so $|A_n| \leq e^{n(h+\varepsilon)}$.
- Proof of (b): Consider the sequence of sets $\Omega_n = \{\omega \in \Omega \mid \mu^n(Z_n(\omega)) \leq e^{-n(h-\varepsilon/2)}\}$. Then $(*)$ implies that $P(\{\Omega_n \text{ eventually }\}) = 1$, i.e. $Z_n \in \Omega_n$ eventually a.s. Consider

the events $[Z_n \in B_n] \cap \Omega_n$. We have:

$$\begin{aligned} P([Z_n \in B_n] \cap \Omega_n) &= \sum_{g \in B_n} \mu^n(g) \mathbf{1}_{[\exists \omega \in \Omega_n, Z_n(\omega)=g]} \\ &\leq |B_n| e^{-n(h-\varepsilon/2)} \\ &\leq e^{n(h-\varepsilon)} e^{-n(h-\varepsilon/2)} \\ &= e^{-n\varepsilon/2} \end{aligned}$$

Thus, $\sum_n P([Z_n \in B_n] \cap \Omega_n)$ converges, so by Borel-Cantelli lemma, we conclude that $P([Z_n \in B_n] \cap \Omega_n \text{ i.o. }) = 0$, so $P([Z_n \in B_n] \cap \Omega_n \text{ eventually }) = 0$. But $P(\{\Omega_n \text{ eventually }\}) = 1$, so $P([Z_n \in B_n] \text{ eventually }) = 0$.

■

Definition 2.8. Suppose (X, d) is a metric space and Γ acts on (X, d) by isometries. The **volume entropy** of Γ in its action on (X, d) is defined to be $v = \text{limsup}_{R \to \infty} \frac{1}{R} |\{g \in \Gamma \mid g \cdot x_0 \in B(x_0, R)\}|$ where $x_0 \in X$.

Proposition 2.8 (Guivarc'h inequality). *Let (X, d) be a metric space and suppose Γ acts on (X, d) by isometries. Consider the random walk $RW_{\Gamma,\mu}$ on Γ. Pick any $x_0 \in X$. Suppose:*

(a) $E[d(x_0, X_1 \cdot x_0)] < \infty$
(b) $v = limsup_{R \to \infty} \frac{1}{R} |\{g \in \Gamma \mid g \cdot x_0 \in B(x_0, R)\}| < \infty$

Then $H(\mu) < \infty$ and $h \leq lv$.

Proof. <u>$H(\mu) < \infty$</u>:

- $H(\mu) = \sum_{g \in \Gamma} -\mu(g) \log(\mu(g)) = E[-\log(\mu)] = \int_0^\infty \mu(\{g \in \Gamma \mid \mu(g) \leq e^{-t}\}) dt$. We break this integral up into two parts as follows.
- Let $a > 0$, to be fixed later. For each $t > 0$ we consider the set $A_{x_0,t} := \{g \in \Gamma \mid \mu(g) \leq e^{-t}\} \cap \{g \in \Gamma \mid g \cdot x_0 \in B(x_0, at)\}$ and $\tilde{A}_{x_0,t} := \{g \in \Gamma \mid \mu(g) \leq e^{-t}\} \cap \{g \in \Gamma \mid g \cdot x_0 \notin B(x_0, at)\}$.

$$\mu(\{g \in \Gamma \mid \mu(g) \leq e^{-t}\}) = \mu(A_{x_0,t}) + \mu(\tilde{A}_{x_0,t})$$

- Fix $\varepsilon > 0$. Then there exists $t_0 > 0$ such that for all $t \geq t_0$, we have $|B(x, t)| \leq e^{(v+\varepsilon)t}$. Therefore, for $t \geq t_0/a$:

$$\begin{aligned} \mu(A_{x,t}) &= \sum_{\substack{g \in \Gamma \\ g \cdot x_0 \in B(x_0, at)}} \mu(g) \mathbf{1}_{[\mu(g) \leq e^{-t}]} \\ &\leq |B(x_0, at)| e^{-t} = e^{(a(v+\varepsilon)-1)t} \end{aligned}$$

We fix a such that $0 < a < 1/(v + \varepsilon)$.

- Now:

$$\begin{aligned} \int_0^\infty \mu(A_{x_0,t}) dt &= \int_0^{t_0/a} \mu(A_{x_0,t}) dt + \int_{t_0/a}^\infty \mu(A_{x_0,t}) dt \\ &\leq \frac{t_0}{a} + \frac{1 - e^{((v+\varepsilon)-a)t_0}}{1 - a(v + \varepsilon)} < \infty \end{aligned}$$

$$\begin{aligned}\int_0^\infty \tilde{A}_{x_0,t}dt &= \int_0^\infty \mu(\{g\in\Gamma \mid \mu(g)\leq e^{-t}, d(x_0,g\cdot x_0)\geq at\})dt\\ &\leq \int_0^\infty \mu(\{g\in\Gamma \mid d(x_0,g\cdot x_0)\geq at\})dt\\ &= E[d(x_0,X_1\cdot x_0)]<\infty\end{aligned}$$

Thus $H(\mu)<\infty$.

$h\leq lv$: Fix $\varepsilon>0$.

(i) $v<\infty$, so there exists a $t_0>0$ such that for all $t\geq t_0$, we have $|\{g\in\Gamma \mid g\cdot x_0\in B(x_0,t)\}|\leq e^{(v+\varepsilon)t}$

(ii) (a) implies that we have $l\in\mathbb{R}_{\geq 0}$ such that for a.e. $\omega\in\Omega$, $\lim\limits_{n\to\infty}\frac{d(x_0,Z_n(\omega)\cdot x_0)}{n}=l$

Consider the sets $B_n=\{g\in\Gamma \mid g\cdot x_0\in B(x_0,(l+\varepsilon)n)\}$. Then (ii) implies that $P(\{[Z_n\in B_n]$ eventually $\})=1$. Using the contrapositive of part (b) in the Zoning lemma, we get: $|B_n|\geq e^{n(h-\varepsilon)}$ for all n. Also using (i), for all $n\geq t_0$, we have $|B_n|\leq e^{n(v+\varepsilon)(l+\varepsilon)}$. So we have $h-\varepsilon\leq lv+(l+v)\varepsilon+\varepsilon^2$. Taking $\varepsilon\to 0$ gives us $h\leq lv$. ∎

3. Hyperbolicity of the Green metric

We begin with the following observation:

Lemma 3.1. *Let Γ be a finitely generated, non-amenable group. Let S be a finite generating set for Γ and d_w be the corresponding word metric on Γ. Consider a symmetric, irreducible random walk $RW_{\Gamma,\mu}$ with finite exponential moment, that is:*

$$\exists\lambda>0, E[e^{\lambda d_w(e,Z_1)}]=E<\infty$$

Then the identity map $(\Gamma,d_G)\xrightarrow{id_\Gamma}(\Gamma,d_w)$ is a quasi-isometry.

Proof. Let $L=\max\{d_G(e,s)\mid s\in S\}$.

Claim 3.1. For all $x\in\Gamma$, $d_G(e,x)\leq Ld_w(e,x)$.

Proof. If $x=s_1\dots s_k$ where $s_1,\dots,s_k\in S$, then:

$$\begin{aligned}d_G(e,x)&\leq\sum_{j=0}^{k-1}d_G(s_1\dots s_j,s_1\dots s_{j+1})\quad\text{(triangle inequality)}\\ &\leq\sum_{j=0}^{k-1}d_G(e,s_{j+1})\quad(\Gamma\text{-invariance of }d_G)\\ &\leq Lk\end{aligned}$$

Therefore, $d_G(e,x)\leq Ld_w(e,x)$. ∎

Claim 3.2. There exist constants $c_1,c_2>0$ such that for all $x\in\Gamma$, we have

$$G(e,x)\leq c_1e^{-c_2d_w(e,x)}$$

Proof. Let $b>0$ and for each $n\in\mathbb{N}_{>0}$ define $Y_n=\sup_{1\leq k\leq n}d_w(e,Z_k)$. Now $y\mapsto e^{\lambda y}$ is a positive, monotonically increasing function, so by Chebyshev's inequality:

$$P[Y_n\geq nb]\leq e^{-n\lambda b}E[e^{\lambda Y_n}]$$

Now note that given $1 \leq k \leq n$, we have:

$$\begin{aligned} d_w(e, Z_k) &\leq \sum_{j=0}^{k-1} d_w(Z_j, Z_{j+1}) \quad \text{(triangle inequality)} \\ &\leq \sum_{j=0}^{k-1} d_w(e, X_{j+1}) \quad (\Gamma\text{-invariance of } d_G) \\ &\leq \sum_{j=0}^{n-1} d_w(e, X_{j+1}) \end{aligned}$$

So $Y_n \leq \sum_{j=0}^{n-1} d_w(e, X_{j+1})$ for all $n \geq 1$. Recall that $\{X_j\}_{j \geq 1}$ is an i.i.d sequence of random variables. So $E[e^{\lambda Y_n}] \leq E^n$. So we have:

$$P[Y_n \geq nb] \leq e^{-n(\lambda b - \log(E))} \quad (*)$$

Choose $b > |\log(E)/\lambda|$ and let $c = \lambda b - \log(E)$. We estimate the Green's function by breaking it up into two pieces:

$$G(e, x) = \sum_{n \geq 0} \mu^n(x) = \sum_{n \leq \frac{|x|}{b}} \mu^n(x) + \sum_{n > \frac{|x|}{b}} \mu^n(x)$$

where, $|x| = d_w(e, x)$. For bounding the first piece we use $(*)$:

$$\begin{aligned} \sum_{n \leq \frac{|x|}{b}} \mu^n(x) &\leq \frac{|x|}{b} \sup\{\mu^n(x) \mid 1 \leq n \leq \frac{|x|}{b}\} \\ &\leq \frac{|x|}{b} P[\exists n \leq \frac{x}{b}, Z_n = x] \\ &\leq \frac{|x|}{b} P[Y_{|x|/b} \geq |x|] \\ &\leq \frac{|x|}{b} e^{-\frac{c}{b}|x|} \leq C' e^{-\frac{c}{2b}|x|} \end{aligned}$$

where $C' > 0$ is chosen large enough so that $|x| \leq bC' e^{c|x|/2b}$. Next, using Corollary 2.1.1:

$$\sum_{n > \frac{|x|}{b}} \mu^n(x) \leq C \sum_{n > \frac{|x|}{b}} e^{-mn} \leq C' e^{-\frac{m}{b}|x|}$$

for large enough $C' > 0$. Now our claim follows easily. ■

Note that Claim 3.2 implies that the Green metric dominates the word metric:

$$d_G(e, x) \geq c_2 d_w(e, x) - \log(c_1)$$

and therefore together with Claim 3.1 and the Γ-invariance of d_G, we have:

$$\forall x, y \in \Gamma, c_2 d_w(e, x) - \log(c_1) \leq d_G(e, x) \leq L d_w(e, x)$$

which implies that id_Γ is indeed a quasi-isometry. ■

Unfortunately, at this point, since we cannot expect (Γ, d_G) to be a geodesic metric space, we cannot immediately conclude in the above lemma that the Green metric is hyperbolic. We ask, when is a metric space which is quasi-isometric to a geodesic hyperbolic metric space, also hyperbolic?

Definition 3.1. Let (X, d) be a metric space. For $\tau \geq 0$ a τ-quasiruler is a quasigeodesic $\alpha : I \to X$ (where I is a connected interval in $\mathbb{R}$), such that for any numbers $s < t < u$ in I, we have $(g(s) \mid g(u))_{g(t)} \leq \tau$. Here $(\cdot|\cdot)_{(\cdot)}$ denotes the Gromov product.
(X, d) is said to be **quasiruled** if there exist constants $\lambda \geq 1, \tau, c > 0$ such that (X, d) is a (λ, c)-quasigeodesic space and every (λ, c)-quasigeodesic is τ-quasiruled.

Theorem 3.1. *Suppose (X, d) is a geodesic hyperbolic metric space, (X', d') is a metric soace and $\varphi : X \to X'$ is a quasi-isometry. Then the following are equivalent:*

(1) X' is hyperbolic.
(2) X' is quasiruled.
*(3) φ is a **ruling**, i.e. there exists a constant $\tau \geq 0$ such that the image of every geodesic segement of X under φ is a τ-quasiruler.*

Proof. ■

Lemma 2.1 and theorem 3 imply:

Proposition 3.1. *Let Γ be a non-elementary hyperbolic group and $RW_{\Gamma,\mu}$ be a symmetric, irreducible random walk with a finite exponential moment. TFAE:*

(1) The map $(\Gamma, d_G) \xrightarrow{id_\Gamma} (\Gamma, d_w)$ is a quasi-isometry and (Γ, d_G) is hyperbolic
*(2) **[Ancona inequality]** For all $r > 0$ there exists $C(r) > 0$ such that $F(x, y) \leq C(r)F(x, z)F(z, y)$ where $x, y \in \Gamma$ and z is any point in Γ within a distance of r from any d_w-geodesic segment joining x to y.*

Theorem 3.2. *Let Γ be a non-elementary hyperbolic group and $RW_{\Gamma,\mu}$ be a symmetric, irreducible random walk such that supp(μ) is finite. $RW_{\Gamma,\mu}$ satisfies the Ancona inequality.*

Together, theorem 4 and proposition 2.1 imply:

Theorem 3.3. *Let Γ be a non-elementary hyperbolic group and $RW_{\Gamma,\mu}$ be a symmetric, irreducible random walk such that supp(μ) is finite. (Γ, d_G) is hyperbolic.*

4. Deviation Inequalities

In this section, we prove a couple of deviation inequalities.

Definition 4.1. Let $((X, d), w)$ be a pointed proper hyperbolic metric space. For $R > 0$, the **shadow** of the ball of radius R centered at a point $x \in X$, is defined to be the set of all points $a \in \partial X$ such that $(a|w)_x \leq R$ or equivalently, $(a|x)_w \geq d(x, w) - R$ and denoted by $S_w(x, R)$.

Proposition 4.1 (Shadows of balls are almost visual balls)**.** *Let (X, d, w) be a pointed hyperbolic space and equip ∂X with a visual metric of parameter $\varepsilon > 0$. Suppose $\tau \geq 0$. There exist constants $C, R_0 > 0$ such that for all $R > R_0, a \in \partial X, x \in X$ such that $(w|a)_x \leq \tau$, we have:*

$$B_\varepsilon(a, \frac{1}{C} e^{R\varepsilon} e^{-\varepsilon d(w,x)}) \subset S_w(x, R) \subset B_\varepsilon(a, C e^{R\varepsilon} e^{-\varepsilon d(w,x)})$$

Lemma 4.1 (Shadow lemma)**.** *Let (X, d, w) be a pointed proper hyperbolic metric space equipped with a geometric action of a non-elementary hyperbolic group Γ. Let d_ε denote a visual metric on ∂X for the parameter $\varepsilon > 0$ and ρ denote the Hausdorff measure on $(\partial X, d_\varepsilon)$ of dimension v/ε where v is the volume entropy. There exists $R_0 > 0$ such that for all $x \in X$ and for all $R \geq R_0$, we have $\rho(S_w(x, R)) \asymp e^{-vd(w,x)}$.*

Martin Boundary and harmonic/hitting measure of a random walk:
Let Γ be a countable discrete group and $\mathrm{RW}_{\Gamma,\mu}$ be a symmetric, irreducible and transient random walk on Γ. Let $C(\Gamma)$ denote the space of continuous real-valued functions on Γ equipped with the topology of pointwise convergence. Consider the map $\Phi : \Gamma \to C(\Gamma)$ defined by:

$$\forall x, y \in \Gamma, \Phi(y)(x) = K_y(x) := \frac{G(x, y)}{G(e, y)}$$

K_y is called a Martin kernel and is in particular a μ-harmonic function:

$$\forall x, y, z \in \Gamma, K_y(x) = \sum_{z \in \Gamma} K_y(z)\mu(x^{-1}z)$$

This follows from the observation that $G(x, y) = \sum_{z\in\Gamma} \mu(x^{-1}z)G(z, y)$.

Claim 4.1. Φ is injective and the closure of $\Phi(\Gamma)$ in $C(\Gamma)$ is compact.

The Martin boundary of $\mathrm{RW}_{\Gamma,\mu}$ is defined to be $\partial_M\Gamma := \overline{\Phi(\Gamma)} - \Phi(\Gamma)$, with the subspace topology. Next, observe that $K_y(x) = \exp(d_G(e, y) - d_G(x, y))$. Thus the Martin boundary coincides with the horofunction boundary of Γ wrt the Green metric. The following is a result due to Kaimanovich from [Kai00].

Theorem 4.1 (V. Kaimanovich)**.** *Let Γ be a finitely generated non-elementary hyperbolic group and let $w \in \Gamma$. Let d_w be the word metric on Γ wrt to some finite generating set of Γ. Consider a symmetric, irredicuble random walk $RW_{\Gamma,\mu}$ on Γ with finite first moment. Then $(Z_n(w))_{n\geq 0}$ almost surely converges to a point $Z_\infty(w)$ on the boundary $\partial\Gamma$.*
For each $a \in \partial\Gamma$, choose a quasigeodesic $[w, a)$ in a measurable way. For each n there is a measurable map $\pi_n : \partial\Gamma \to \Gamma$ such that $\pi_n(a) \in [w, a)$ and for almost every trajectory of the random walk,

$$\lim_{n\to\infty} \frac{d_w(Z_n(w), \pi_n(Z_\infty(w)))}{n} = 0$$

Given any $a \in \partial\Gamma$, the sequence $(\pi_n(a))_n$ is a discrete approximation of the quasigeodesic ray $[w, a)$ so the theorem tells us that almost surely, $d_w(Z_n(w), Z_\infty(w)) = o(n)$, i.e a random sample path is **sublinearly tracked by quasigeodesic rays**. We will see more about this in the coming sections.
Under the assumptions of theorem 6, we define the hitting measure ν on $\partial\Gamma$ as $\nu_w(A) = P[Z_\infty(w) \in A]$ for every Borel-measurable set A in $\partial\Gamma$.

Theorem 4.2. *Let Γ be a finitely generated non-elementary hyperbolic group and $RW_{\Gamma,\mu}$ be a symmetric, irreducible random walk on Γ such that (Γ, d_G) is hyperbolic and quasi-isometric to (Γ, d) where d is the word metric wrt some finite generating set. Equip $\partial\Gamma$ with a visual metric $d_{G,\varepsilon}$ wrt to the Green metric for $\varepsilon > 0$ small enough. Then the hitting measure ν is Ahlfors-regular of dimension $1/\varepsilon$ and in fact, the volume entropy of Γ with respect to the Green metric is equal to 1.*

Proposition 4.2. *Let Γ be a finitely generated non-elementary hyperbolic group $w \in \Gamma$. Let d be the word metric on Γ wrt to some finite generating set. Let $RW_{\Gamma,\mu}$ be a symmetic, irreducible random walk on Γ such that (Γ, d_G) is hyperbolic and quasi-isometric to (Γ, d). For each $a \in \partial\Gamma$, choose a quasigeodesic $[w, a)$ in a measurable way. There exists $b > 0$ such that for all $D > 0$ and $n \in \mathbb{N}_{>0}$,*

$$P[d(Z_n(w), [w, Z_\infty(w))) \geq D] \lesssim e^{-bD}$$

Proof. Before we begin, let us note that for $g, z \in \Gamma$ if $z = gw$ then $Z_\infty(z) = gZ_\infty(w)$ and that the sequence $(d(Z_n(w), [w, Z_\infty(w))))$ records the lateral deviation of the random walk $(Z_n(w))$ from the quasi-geodesic $[w, Z_\infty(w))$.

$$\begin{aligned} P[d(Z_n(w), [w, Z_\infty(w))) \geq D] &= \sum_{z\in\Gamma} P[d(Z_n(w), [w, Z_\infty(w))) \geq D, Z_n(w) = z] \\ &= \sum_{z\in\Gamma} P[d(z, [w, Z_n^{-1} Z_\infty(z))) \geq D, Z_n(w) = z] \\ &= \sum_{z\in\Gamma} P[d(z, [w, Z_n^{-1} Z_\infty(z))) \geq D] P[Z_n(w) = z] \\ &= \sum_{z\in\Gamma} P[d(z, [w, Z_\infty(z))) \geq D] P[Z_n(w) = z] \end{aligned}$$

where additionally we have used the observation that $Z_n^{-1}Z_\infty$ and Z_n are independent random variables and that $Z_n^{-1}Z_\infty \stackrel{d}{=} Z_\infty$. Now fix $z \in \Gamma$ and consider a sample path $Z_n(w)$ such that $d(z, [w, Z_\infty(z))) \geq D$. In particular, we have $d(z, w) \geq D$. Let x be the midpoint of a geodesic segment $[z, w]$. Since the triangle $(x, z, Z_\infty(z))$ is δ-thin (for some $\delta \geq 0$), we have $(z|Z_\infty(z))_x \leq \delta$. That is, $Z_\infty(z) \in S_z(x, R)$ where $R > \delta$. So we have:

$$P[d(z, [w, Z_\infty(z))) \geq D] \leq P[Z_\infty(z) \in S_z(x, R)] = \nu_z(S_z(x, R))$$

Now since (Γ, d_G) is non-elementary hyperbolic, using Theorem 4.2 and the Shadow lemma we have $\nu_z(S_z(x, R)) \leq e^{-d_G(z,x)}$ where R is a universal constant. Using the quasi-isometry between (Γ, d_G) and (Γ, d), we get $P[d(z, [w, Z_\infty(z))) \geq D] \lesssim e^{-2bd(z,x)} \leq e^{-bD}$ for some constant $b > 0$. Therefore:

$$\begin{aligned} P[d(Z_n(w), [w, Z_\infty(w))) \geq D] &= \sum_{z\in\Gamma} P[d(z, [w, Z_\infty(z))) \geq D] P[Z_n(w) = z] \\ &\lesssim \sum_{z\in\Gamma} e^{-bD} P[Z_n(w) = z] = e^{-bD} \end{aligned}$$

■

Corollary 4.2.1. *Under the assumptions of the proposition above, we have:*

$$\limsup_{n\to\infty} \frac{d(Z_n(w), [w, Z_\infty(w)))}{\log n} < \infty \ a.s.$$

Proof. Note that $P[d(Z_n(w), [w, Z_\infty(w))) \geq \frac{2}{b} \log n] \lesssim 1/n^2$. Now the claim follows using the Borel-Cantelli lemma. ■

5. Dimension of the harmonic measure on the boundary of a hyperbolic metric space

The **dimension** of a measure is defined to be the infimum of the Hausdorff dimensions of sets of positive meausure. We first define the shadow of a ball in an alternative, perhaps more blatantly geometric way. Essentially, our definition of the shadow of a ball $B(x, R)$ in a hyperbolic metric space X seen from a point w, suggests that it *almost* comprises of those points $a \in \partial X$ such that some quasi-geodesic $[w, a)$ intersects $B(x, R)$. We make this precise as follows:

Definition 5.1. Let (X, d) be a hyperbolic metric space equipped with a quasiruling structure $\mathcal{G}$. We say that $\mathcal{G}$ is a **visual quasiruling structure** if any pair of points in $X \cup \partial X$ can be joined by a quasiruler in $\mathcal{G}$. In this case, we define the $\mathcal{G}$-shadow of a ball of radius R centered at $x \in X$ by:

$$S(x, R; \mathcal{G}) = \{a \in \partial X \mid \exists \text{ a quasiruler } [w, a) \in \mathcal{G}, [w, a) \cap B(x, R) \neq \emptyset\}$$

Next we observe that $\mathcal{G}$-shadows are almost shadows.

Proposition 5.1. *Let (X, d) be a hyperbolic space equipped with a visual quasiruling structure $\mathcal{G}$. There exist constants $C, R_0 > 0$ such that for all $R > R_0$, $a \in \partial X$, $w \in X$, quasiruler $[w, a) \in \mathcal{G}$ and $x \in [w, a)$, we have:*

$$S(x, R - C; \mathcal{G}) \subset S_w(x, R) \subset S(x, R + C; \mathcal{G})$$

Proposition 5.2 (Doubling property of harmonic measures)**.** *Let Γ be a finitely generated, non-elementary hyperbolic group. Let $RW_{\Gamma,\mu}$ be a symmetric, irreducible random walk such that:*

- *(Γ, d_G) is hyperbolic and the identity map $(\Gamma, d_G) \overset{id_\Gamma}{\to} (\Gamma, d)$ is a quasi-isometry, where d is the word metric wrt some finite generating set. We will refer to the corresponding cayley (metric) graph by (X, d).*

Let ν be the harmonic measure on ∂X seen from e and equip ∂X with a visual metric of parameter $\varepsilon > 0$. ν has the doubling property, i.e there exists a constant $C.0$ such that if B is any visual ball and $2B$ is the concentric visual ball having twice the radius as B, then $\nu(2B) \leq C\nu(B)$.

Proposition 5.3. *Let Γ be a finitely generated, non-elementary hyperbolic group. Let $RW_{\Gamma,\mu}$ be a symmetric, irreducible random walk such that:*

- *(Γ, d_G) is hyperbolic and the identity map $(\Gamma, d_G) \overset{id_\Gamma}{\to} (\Gamma, d)$ is a quasi-isometry, where d is the word metric wrt some finite generating set. We will refer to the corresponding cayley (metric) graph by (X, d).*
- *$RW_{\Gamma,\mu}$ has finite moment for the Green metric, i.e.*

$$E[d_G(e, Z_1)] < \infty$$

Let ν be the harmonic measure on ∂X seen from e and equip ∂X with a visual metric of parameter $\varepsilon > 0$. For ν-a.e. point $a \in \partial X$, we have:

$$\lim_{r \to \infty} \frac{\log \nu(B_\varepsilon(a, r))}{\log r} = \frac{l_G}{\varepsilon l} \quad (*)$$

where l_G is the Green speed, while l is the drift.

We give a heuristic proof first:

- Note that $(*)$ is equivalent to showing that for P-a.e. $\omega \in \Omega$, we have:
$$\lim_{r\to\infty} \frac{\log \nu(B_\varepsilon(Z_\infty(\omega), r))}{\log r} = \frac{l_G}{\varepsilon l}$$
We attempt to compute the limit above along the sequence $r_n = e^{-\varepsilon n l}$.
- Now pick any $\omega \in \Omega$ such that $\lim_{n\to\infty} d(e, Z_n(\omega))/n = l$. So for large enough n, our random walk is almost a distance nl away from home: $d(e, Z_n(\omega)) \approx nl$. Now we know that visual balls are *almost* shadows of balls. Together with the doubling property of the harmonic measure we get:
$$\nu(B_\varepsilon(Z_\infty(\omega), e^{-\varepsilon d(e,Z_n(\omega))})) \asymp \nu(S(Z_n(\omega), R))$$
where R is some large enough fixed constant. Using the Shadow lemma and the fact that the identity map between (Γ, d) and (Γ, d_G) is a quasi-isometry, we get:
$$\nu(B_\varepsilon(Z_\infty(\omega), e^{-\varepsilon n l})) \asymp e^{-d_G(Z_n(\omega), e)}$$
- Therefore:
$$\lim_{r\to\infty} \frac{\log \nu(B_\varepsilon(Z_\infty(\omega), r))}{\log r} = \lim_{n\to\infty} \frac{d_G(Z_n(\omega), e)}{n\varepsilon l} = \frac{l_G}{\varepsilon l}$$

Now we begin the formal proof:

Proof. Note that Γ is sitting inside X, as a set: namely each group element is sitting at the corresponding vertex in the Cayley graph X. And this inclusion is a quasi-isometry when Γ is equipped with either the word metric or the Green metric.

$$\begin{array}{ccc} (\Gamma, d_G) & \hookrightarrow & (X, d) \\ \Big\| {\scriptstyle id_\Gamma} & \nearrow & \\ (\Gamma, d_w) & & \end{array}$$

Let $\mathcal{G}$ be the set of all geodesics (rays, lines and segments) in X. $\mathcal{G}$ induces a visual quasiruling structure $\mathcal{G}'$ on (Γ, d_G). For any $x \in \Gamma$ and $R > 0$ the sets $S(x, R; \mathcal{G}')$ and $S(x, R; \mathcal{G})$ are identified under the homeomorphism induced by $(\Gamma, d_G) \hookrightarrow (X, d)$ on the respective boundaries. We claim that there exist constants $C, R > 0$ such that for all $a \in \partial X$, for all $x \in [e, a) \in \mathcal{G}$:

$$B_\varepsilon(a, \frac{1}{C} e^{-\varepsilon d(e,x)}) \subset S(x, R; \mathcal{G}) \subset B_\varepsilon(a, C e^{-\varepsilon d(e,x)}) \tag{1}$$

$$\nu(S(x, R; \mathcal{G})) \asymp e^{-d_G(e,x)} \tag{2}$$

Here (1) is deduced from Proposition 4.1 for (X, d). While (2) follows from Proposition 4.1 for $(\Gamma, d_G, \mathcal{G}')$ together with the underlined observation above and the shadow lemma, i.e. Lemma 4.1 for (Γ, d_G). In order to use the shadow lemma, we have also used the fact that ν is a quasi-conformal measure on ∂X and that the volume entropy of Γ wrt the Green metric is equal to 1 (Theorem 4.2).

Next, observe that (1) and the doubling property of ν wrt to the visual metric d_ε on ∂X gives:

$$\forall a \in \partial X, \forall x \in [e, a) \in \mathcal{G}, \nu(B_\varepsilon(a, e^{-\varepsilon d(e,x)})) \asymp \nu(S(x, R; \mathcal{G})) \tag{3}$$

We now implement Kaimanovich's theorem 4.1 for (X, d).

- Firstly, we can assign a geodesic ray $[e, a) \in \mathcal{G}$ to each point $a \in \partial X$ in a measurable way.
- For each n, there exists a measurable "projection" map $\pi_n : \partial X \to X$ such that $\pi_n(a) \in [e, a)$ for all $a \in \partial X$ and:

$$\text{for a.e. } \omega \in \Omega, \lim_{n\to\infty} \frac{d(Z_n(\omega), \pi_n(Z_\infty(\omega)))}{n} = 0 \tag{4}$$

Since $(\Gamma, d_G) \hookrightarrow (X, d)$ is a quasi-isometry, we also have:

$$\text{for a.e. } \omega \in \Omega, \lim_{n\to\infty} \frac{d_G(Z_n(\omega), \pi_n(Z_\infty(\omega)))}{n} = 0 \tag{5}$$

Also recall, that $E[d(e, Z_1)], E[d_G(e, Z_1)] < \infty$ implies that:

$$\text{for a.e. } \omega \in \Omega, \lim_{n\to\infty} \frac{d(e, Z_n(\omega))}{n} = l, \lim_{n\to\infty} \frac{d_G(e, Z_n(\omega))}{n} = l_G \tag{6}$$

Fix $\eta > 0$. Using $(4) - (6)$ let $\Omega' \subset \Omega$ be a full measure set such that for all $\omega \in \Omega'$ there exists $n_0 = n_0(\omega) \in \mathbb{N}$ such that for $n \geq n_0$, we have:

$$|d(e, Z_n(\omega) - nl)| \leq \eta n \qquad d(Z_n(\omega), \pi_n(Z_\infty(\omega))) \leq \eta n$$
$$|d_G(e, Z_n(\omega) - nl_G)| \leq \eta n \qquad d_G(Z_n(\omega), \pi_n(Z_\infty(\omega))) \leq \eta n$$

and therefore, in particular:

$$|d(e, \pi_n(Z_\infty(\omega))) - nl| \leq 2\eta n \quad |d_G(e, \pi_n(Z_\infty(\omega))) - nl_G| \leq 2\eta n \tag{7}$$

Define $r_n = e^{-\varepsilon d(e, \pi_n(Z_\infty(\omega)))}$ and set $a = Z_\infty(\omega), x = \pi_n(Z_\infty(\omega))$ in $(2) - (3)$ to obtain:

$$\nu(B_\varepsilon(Z_\infty(\omega), r_n)) \asymp e^{-d_G(e, \pi_n(Z_\infty(\omega)))}$$

Together with (7):

$$\forall \omega \in \Omega' \exists n_0 = n_0(\omega) \in \mathbb{N}, \text{ for all } n \geq n_0, \left| \frac{\log \nu(B_\varepsilon(Z_\infty(\omega), r_n))}{\log r_n} - \frac{l_G}{\varepsilon l} \right| \lesssim \eta \tag{8}$$

Claim. The fact that the harmonic measure ν has the doubling property on $(\partial X, d_\varepsilon)$ implies that ν is α-homogeneous for some $\alpha > 0$, i.e

$$\exists C > 0 \text{ such that for any } 0 < R_1 < R_2 < \operatorname{diam}(\partial X, d_\varepsilon), \frac{\nu(B_\varepsilon(a, R_2))}{\nu(B_\varepsilon(a, R_1))} \leq C \left(\frac{R_2}{R_1} \right)^\alpha \tag{9}$$

Taking log on both sides of (9), setting $a = Z_\infty(\omega)$, $R_1 = r_n$ and $R_2 = e^{-\varepsilon nl}$ we have:

$$|\log \nu(B_\varepsilon(Z_\infty(\omega), e^{-\varepsilon nl})) - \log \nu(B_\varepsilon(Z_\infty(\omega), r_n)| \leq 2\alpha\varepsilon\eta n + O(1)$$

$$\text{and so } \left| \frac{\log \nu(B_\varepsilon(Z_\infty(\omega), e^{-\varepsilon nl}))}{\log e^{-\varepsilon nl}} - \frac{\log \nu(B_\varepsilon(Z_\infty(\omega), r_n)}{\log r_n} \right| \lesssim \eta + O(1/n)$$

where $\omega \in \Omega', n \geq n_0(\omega)$. Therefore:

$$\limsup_{n\to\infty} \left| \frac{\log \nu(B_\varepsilon(Z_\infty(\omega), e^{-\varepsilon nl}))}{\log e^{-\varepsilon nl}} - \frac{\log \nu(B_\varepsilon(Z_\infty(\omega), r_n)}{\log r_n} \right| \lesssim \eta$$

$$\text{as } \eta > 0 \text{ was arbitrary}, \limsup_{n\to\infty} \left| \frac{\log \nu(B_\varepsilon(Z_\infty(\omega), e^{-\varepsilon nl}))}{\log e^{-\varepsilon nl}} - \frac{\log \nu(B_\varepsilon(Z_\infty(\omega), r_n)}{\log r_n} \right| = 0$$

$$\text{Finally, } \lim_{r\to 0} \frac{\log \nu(B_\varepsilon(Z_\infty(\omega), r))}{\log r} = \lim_{n\to\infty} \frac{\log \nu(B_\varepsilon(Z_\infty(\omega), e^{-\varepsilon n l}))}{loge^{-\varepsilon nl}}$$
$$= \lim_{n\to\infty} \frac{\log \nu(B_\varepsilon(Z_\infty(\omega), r_n))}{r_n}$$
$$= \frac{l_G}{\varepsilon l} \quad \text{(using (8)) for all } \omega \in \Omega'$$

■

Corollary 5.0.1. *ν has dimension $l_G/\varepsilon l$.*

6. The Poisson boundary

Here we take another step towards understanding the asymptotic behaviour of a random walk. To begin with, consider a **transient** random walk $\mathrm{RW}_{\Gamma,\mu}$ on a countable discrete group Γ where the law of the steps is given by the Borel probability measure μ. Since the random walk is transient, almost every sample path $(x_n)_{n\geq 0}$, escapes to infinity. More precisely, define the **positive hitting probabilities**: $\rho^k_{x,A} = P[\tau^k_{x,A} < \infty]$ and $\rho^k_{xy} = P[\tau^k_{xy} < \infty]$ for all $k \geq 0$, $x, y \in \Gamma$, $A \subset \Gamma$. We write $\rho^1_{x,A} = \rho_{x,A}$ and $\rho^1_{xy} = \rho_{xy}$. Then:

Claim. $\forall x, y \in \Gamma$ we have $\rho^k_{xy} = \rho_{xy}\rho^{k-1}_{yy}$.

Now let K be any finite subset of Γ. Let $K_n = [\tau^n_{x,K} < \infty]$ be the measurable set of sample paths that visit K at least n times. We wish to show that $P[K^c_n \text{ eventually}] = 1$ that is, P-almost surely, every sequence leaves the set K. It follows from the claim above that $\rho^n_{x,K} \leq |K|\rho^{n-1}_K$ where $\rho_K = \min\{\rho_{yy} \mid y \in K\}$. The random walk is transient so $\rho_K < 1$ and an application of Borel-Cantelli lemma gives us $P[K_n \text{ i.o.}] = 0$. So $P[K^c_n \text{ eventually}] = 1$.

Now that we have established that almost every sample path $(x_n)_{n\geq 0}$, escapes to infinity, we want to ask: **where does the random walk escape to**? Well, ideally to a "boundary". And if there is none, we must construct it. This leads us to the Poisson boundary. There are multiple levels in which this question can be answered. We will look at the following:

(a) For the random walk $\mathrm{RW}_{\Gamma,\mu}$ we will quickly define the Poisson boundary with the help of the stationary σ-algebra.
(b) When Γ can be equipped with a metric so that it is a complete separable metric space, we will try to construct the Poisson boundary as a so-called compactification boundary.
(c) In this most general case, we will look at irreducible Markov chains on countable state space and define the Poisson boundary.

6.1. **The Poisson boundary of a random walk on a countable group.**

Consider a random walk on Γ, $\mathrm{RW}_{\Gamma,\mu}$. We equip the space of sample paths $G^{\mathbb{N}}$ with the product σ-algebra, generated by cylinder sets. Recall that we have the increment space $\Omega = \Gamma^{\mathbb{N}_{>0}}$ equipped with the product σ-algebra and the product probability measure $P = \mu^{\otimes \mathbb{N}_{>0}}$. We have the map $A : \Gamma \times \Omega \to \Gamma^{\mathbb{N}}$ defined by $A(g, \omega) = (\omega_n)_{\geq 1} = (Z_n(\omega))_{n\geq 0}$. Thus, given a probability measure θ on G, the random walk starting with θ as the initial distribution has law $P_\theta = A_*(\theta \otimes P)$. $P_x = P_{\delta_x}$ is the law of the random walk starting at $x \in \Gamma$. Consider the *time shift* map $T : \Gamma^{\mathbb{N}} \to \Gamma^{\mathbb{N}}$ mapping $(\omega_n)_{n\geq 0}$ to $(\omega_{n+1})_{n\geq 0}$. T is a measure-preserving map on $(\Gamma^{\mathbb{N}}, \mathcal{B}_{\Gamma^{\mathbb{N}}}, P_o)$ where o is any fixed base-point. Let $\mathcal{I} = \{A \in \mathcal{B}_{\Gamma^{\mathbb{N}}} \mid T^{-1}A = A\}$ be the σ-algebra of T-invariant measurable sets.

- Since $(\Gamma^{\mathbb{N}}, \mathcal{B}_{\Gamma^{\mathbb{N}}}, P_o)$ is a standard probability space, there exists a unique (upto measurable isomorphism) measurable space $(\partial_\mu\Gamma, \mathcal{A})$ and a measurable map $b : \Gamma^{\mathbb{N}} \to \partial_\mu\Gamma$ such that $b^{-1}\mathcal{A} = \mathcal{I}(\text{mod} P_o)$. We claim that $\partial_\mu\Gamma$ is in fact the space of ergodic components.
- Since $(\Gamma^{\mathbb{N}}, \mathcal{B}_{\Gamma^{\mathbb{N}}}, P_o)$ is a standard probability space, the Ergodic decomposition theorem gives us the following:
  - A disjoint collection $\mathcal{P}$ of measurable subsets of $\Gamma^{\mathbb{N}}$ such that $\cup_{P\in\mathcal{P}}P$ has full measure.
  - Equip $\mathcal{P}$ with the quotient σ-algebra. We have a collection of probability measures $(\mu_P)_{p\in\mathcal{P}}$ and a probability measure $\hat{\mu}$ on $\mathcal{P}$ such that for all $A \in \mathcal{B}_{\Gamma^{\mathbb{N}}}$ the map $P \mapsto \mu_P(A)$ is measurable and
  $$\forall A \in \mathcal{B}_{\Gamma^{\mathbb{N}}}, P_o(A) = \int_{\mathcal{P}} \mu_P(A) d\hat{\mu}(P)$$
  and each μ_P is T-invariant and ergodic.
  - For $\hat{\mu}$-a.e. $P \in \mathcal{P}$, $\mu_P(P) = 1$.

  In the above, we can in fact choose each $P \in \mathcal{P}$ to be T-invariant. We can check that $\mathcal{P}$ generates the σ-algebra $\mathcal{I}$ mod P_o.
- Thus we can take $\partial_\mu\Gamma$ to be the measurable space of ergodic components and b to be the projection map (defined almost everwhere). Note that by construction, $b = b \circ T$ almost everywhere.

We call $\partial_\mu\Gamma$ the **Poisson boundary** of Γ for the random walk $\text{RW}_{\Gamma,\mu}$. It comes equipped with a family of **harmonic measures** $(\nu_x)_{x\in\Gamma}$ where $\nu_x = b_*P_x$. Note that $P_x = \sum_{y\in\Gamma}\mu(x^{-1}y)P_y$ and so $\nu_x = \sum_{y\in\Gamma}\mu(x^{-1}y)\nu_y$, for all $x \in \Gamma$. While the Poisson boundary as defined above, does seem to be record the *limiting* futures of sample paths, it in fact also solves the Dirichlet problem for harmonic functions defined wrt to transition probabilities $p(x, y) = \mu(x^{-1}y)$ on Γ.

6.2. **The Poisson formula and Poisson boundary for Markov chains.**

Let us recall the Dirichlet problem for the unit disc $\mathbb{D}$. Suppose we are given a continuous function on the boundary $\partial\mathbb{D}$ of the unit disc. Can we extend it continuously to a harmonic function on $\mathbb{D}$? This is the Dirichlet problem, and it is solvable for the unit disc: For every $\hat{f} \in C(\partial\mathbb{D})$, there exists a unique function $f \in C(\bar{\mathbb{D}})$ such that $f\big|_{\partial\mathbb{D}} = \hat{f}$ and $f\big|_{\mathbb{D}}$ is harmonic. So, for each point $x \in \mathbb{D}$ we have a positive linear functional on $C(\partial\mathbb{D})$ given by $\hat{f} \mapsto f(x)$. By Riesz-Markov representation theorem, there is a Radon measure ν_x on $\partial\mathbb{D}$ such that $f(x) = \int_{\partial\mathbb{D}} \hat{f}(y)d\nu_x(y)$. This is essentially the Poisson formula. $x \mapsto \nu_x$ embeds $\mathbb{D}$ into $\mathcal{M}_1(\partial\mathbb{D})$, the space of Borel probability measures on $\partial\mathbb{D}$. Let us make some observations:

- For every conformal automorphism $g \in \text{Aut}(\mathbb{D})$, $g_*\nu_x = \nu_{g(x)}$.
- Since f is harmonic, it follows that $(\nu_x)_{x\in\mathbb{D}}$ is a family of harmonic measures.
- In particular, ν_0 is invariant under circle rotations so it must be the Lebesgue measure on $\partial\mathbb{D}$.

- In fact:

$$\begin{aligned} f(x) &= \int_{\partial\mathbb{D}} \hat{f}(y)d\nu_x(y) \\ &= \int_{\partial\mathbb{D}} \hat{f}(y)(d\nu_x/d\nu_o)(y)d\nu_0(y) \\ &= \int_0^{2\pi} \hat{f}(e^{i\theta})\Pi(x,y)d\theta/2\pi \end{aligned}$$

  where $\Pi(x,y) = (1-|x|^2)/|y-x|^2$ is the so-called Poisson kernel. This Poisson kernel has a probabilistic interpretation. $d\nu_x$ is the hitting measure on $\partial\mathbb{D}$ for Brownian motion initiated at x.
- On the other hand, given a bounded harmonic function u defined on $\mathbb{D}$, it extends continuously to $\bar{\mathbb{D}}$. One simply takes radial limits: $u(\xi) = \lim_{r\to 1^-} u(r\xi)$ where $\xi \in \partial\mathbb{D}$.
- Moreover, since $\lim_{x\to\xi} d\nu_x = \delta_\xi$, we get a **linear isometric isomorphism** $C(\partial\mathbb{D}) \to H^\infty(\mathbb{D})$ ($\hat{f} \mapsto u$ as above) between the Banach spaces of continuous functions on $\partial\mathbb{D}$ and that of boundaed harmonic functions on $\mathbb{D}$.

Let us now look at a discrete analogue of this phenomena. We consider a Markov chain on a countable state space V with transition matrix $P = (p(x,y))_{x,y\in V}$. Consider the sequence space $V^{\mathbb{N}}$ equipped with the product σ-algebra. Let $X_n : V^{\mathbb{N}} \to V$ be the projection onto n^{th} coordinate and let P_θ denote the law of the Markov chain starting with initial distribution θ. That is:

- Let $P_\theta^{(0)} = \theta$. And for $n \geq 1$, define the probability measures $P_\theta^{(n)}$ on V^n as follows: For subsets $(B_i)_{0\leq i\leq n}$ of V,

$$P_\theta^{(n)}(X_j \in B_j; 0 \leq j \leq n) = \int_{B_0} d\theta(x_0) \int_{B_1} p(x_0, dx_1) \cdots \int_{B_n} p(x_{n-1}, dx_n)$$

  $(P_\theta^{(n)})$ are consistent as marginals and glue to give the probability measure P_θ on $V^{\mathbb{N}}$.
- So that (X_n) is a Markov chain with transition matrix P, that is, $P_\theta(X_{n+1} = y \mid X_0, \ldots, X_n) = p(X_n, y)$ for all $y \in V$.
- Note that just as in the previous subsection, we have the *time shift* map $T : V^{\mathbb{N}} \to V^{\mathbb{N}}$. Thus we also have the stationary σ-algebra $\mathcal{I}$ of T-invariant Borel sets. An $\mathcal{I}$-measurable function will be called a stationary function. Note that a Borel measurable function is stationary iff $f = f \circ T$.

The transition matrix defines a natural averaging operator, the so-called Markov operator. Given a function $f : V \to \mathbb{R}$, define $Pf(x) = \sum_{y\in V} p(x,y)f(y)$. f is said to be P-harmonic if $Pf = f$. Now that we have harmonic functions on V, can we hope to find a "boundary" and a corresponding Poisson formula for the P-harmonic functions on V?

- Let $L^\infty(V^{\mathbb{N}}, \mathcal{I}, P_o)$ be the Banach space of P_o-equivalence classes of bounded stationary functions on $V^{\mathbb{N}}$. Recall that $f, g : V \to \mathbb{R}$ are P_o-equivalent if $P_o[f \neq g] = 0$.
- Let $H^\infty(V, P)$ be the Banach space of bounded P-harmonic functions on V.

Proposition 6.1 (Poisson Formula I)**.** *Suppose (V,P) is irreducible, then we have a linear isometric isomorphism $\Phi : L^\infty(V^{\mathbb{N}}, \mathcal{I}, P_o) \to H^\infty(V,P)$ mapping f to U_f for each bounded stationary function f, where $U_f(x) = E_x[f(X_0, X_1, \ldots)]$ for all $x \in V$.*

Proof.

- For every bounded stationary function f, U_f is bounded and P-harmonic Since f is bounded, we have $||U_f||_\infty \le ||f||_\infty$, so U_f is bounded. Once we've showed that Φ is well-defined, this will also show that it is a **contraction**. Next, observe that:

$$\begin{aligned} E_x[f] &= E_x[E_x[f \mid X_0]] \\ &= E_x[E_x[f \circ T \mid X_0]] \quad (f = f \circ T) \\ &= E_x[E_{X_1}[f]] \quad \text{(Markov property)} \\ &= \sum_{y \in V} p(x,y) E_y[f] \end{aligned}$$

- Φ is well-defined Take two bounded stationary functions f, g which are P_o-equivalent. We wish to show that $U_f = U_g$. Let $x \in V$.

$$\forall x \in V, U_f(x) - U_g(x) = E_x[f - g] \le M P_x[f \neq g]$$

where $M > 0$ is some large constant. But P_x is absolutely continuous wrt P_o: this follows the fact that for some large enough n, $p^n(o,x) > 0$ and $P_o = \sum_{y \in V} p^n(o,y) P_y$ so $P_o \ge p^n(o,x) P_x$. Therefore $P_x[f \neq g] = 0$. As x was arbitrary, $U_f = U_g$.
- Constructing the inverse Define $\Psi : H^\infty(V,P) \to L^\infty(V^{\mathbb{N}}, \mathcal{I}, P_o)$ by $\Psi(u) = F_u$ where $F_u((x_n)_{n \ge 0}) = \limsup_{n \to \infty} u(x_n)$. Note that Ψ is clearly well-defined, linear and a contraction. We now check that $\Psi \circ \Phi = \mathrm{id}_{L^\infty(V^{\mathbb{N}}, \mathcal{I}, P_o)}$ and $\Phi \circ \Psi = \mathrm{id}_{H^\infty(V,P)}$ and this will complete the proof.

$\Psi \circ \Phi = \mathrm{id}_{L^\infty(V^{\mathbb{N}}, \mathcal{I}, P_o)}$: Let f be any bounded stationary function on V. We need to show that F_{U_f} is P_o-equivalent to f. Let us evaluate F_{U_f}.

$$\begin{aligned} F_{U_f}((X_n)) &= \limsup_{n \to \infty} U_f(X_n) \\ &= \limsup_{n \to \infty} E_{X_n}[f] \\ &= \limsup_{n \to \infty} E_o[f \circ T^n | X_0, \ldots, X_n] \quad \text{(Markov Property)} \\ &= \limsup_{n \to \infty} E_o[f(X_0, X_1, \ldots) | X_0, \ldots, X_n] \quad (f \text{ is stationary}) \\ &= \lim_{n \to \infty} E_o[f(X_0, X_1, \ldots) | X_0, \ldots, X_n] \text{ a.s.} \quad \text{(Martingale convergence theorem)} \\ &= f(X_0, X_1, \ldots) \text{ a.s.} \end{aligned}$$

$\Phi \circ \Psi = \mathrm{id}_{H^\infty(V,P)}$: Let u be any bounded harmonic function. We wish to show that $U_{F_u} = u$. Let us evaluate U_{F_u} at a point $x \in V$.

$$\begin{aligned} U_{F_u}(x) &= E_x[F_u(X_0, X_1, \ldots)] \\ &= E_x[\limsup_{n \to \infty} u(X_n)] \end{aligned}$$

Since u is harmonic, $(u(X_n))$ is a martingale. So by the Martingale convergence theorem, $(u(X_n))$ converges a.s. and in L^1. So $U_{F_u}(x) = \lim_{n \to \infty} E_x[u(X_n)]$. But $E_x[u(X_n)] = \sum_{y \in V} P_x[X_n = y] u(y) = u(x)$ as u is harmonic.

■

Proposition 6.1 suggests that the space $(V^{\mathbb{N}}, \mathcal{I})$ is a candidate for the boundary we were looking for. However, it is much too big. For example, for a simple random walk on a regular tree of degree at least 3, we see that we would like to identify two sample paths that converge to the same end.

Definition 6.1. A **measure-theoretic boundary** of an irreducible Markov chain on a countable state space V, transition matrix P, is a measurable space $(B, \mathcal{F}_B)$ together with a measurable map $b : (V^{\mathbb{N}}, \mathcal{B}_{V^{\mathbb{N}}}) \to (B, \mathcal{F}_B)$ such that:

- b is T-invariant: $b = b \circ T$.
- $\mathcal{F}_B$ is countably generated and separates points of B: There is a sequence of measurable sets (A_n) in $\mathcal{F}_B$ such that $\mathcal{F}_B = \sigma(A_1, A_2, \dots)$ and for every distinct pair $x, y \in B$, $1_{A_n}(x) \neq 1_{A_n}(y)$ for some n.

We equip this boundary with a family of **harmonic measures**: $\nu_x = b_*(P_x)$ for each $x \in V$. Irreducibilty implies that these measures are absolutely continuous wrt to each other and the Radon-Nikodym derivatives $d\nu_x/d\nu_o(y)$ serve as Poisson kernels, just as in the classical Poisson formula we discussed above.

Remark. Note that the random walk $\mathrm{RW}_{\Gamma,\mu}$ on Γ is a Markov chain with state space $V = \Gamma$ and $p(x, y) = \mu(x^{-1}y)$ for all $x, y \in \Gamma$. The Poisson boundary $(\partial_\mu\Gamma, \mathcal{A}, b)$ is a measure-theoretic boundary.

Once we have a measure-theoretic boundary, the harmonic measures yield one half of solution to the Dirichlet problem.

Proposition 6.2. *Suppose that $(B, \mathcal{F}_B, b)$ is a boundary for the irreducible Markov chain (V, P). Given any bounded measurable function φ on B, we have a "continuous" extension u of φ to $V \cup B$: Define $u : V \to \mathbb{R}$ by $u(x) = \int_B \varphi(\theta) d\nu_x(\theta)$. We have:*

(i) u is a bounded harmonic function.
(ii) $\lim_{n\to\infty} u(X_n) = \varphi(b((X_n)_{n\geq 0}))$ P_o-a.s. for all $o \in V$.

Proof. Note that $f := \varphi \circ b \in L^\infty(V^{\mathbb{N}}, \mathcal{I}, P_o)$. Following the notation introduced in proof of proposition 6.1:

$$\begin{aligned} U_f(x) = E_x[f] &= \int_{V^{\mathbb{N}}} \varphi \circ b dP_x \\ &= \int_B \varphi d\nu_x = u(x) \end{aligned}$$

So $\lim_{n\to\infty} u(X_n) = \lim_{n\to\infty} U_f(X_n)$. But using Levy's $0-1$ law, we have $\lim_{n\to\infty} U_f(X_n) = f((X_n)_{n\geq 0}) = \varphi(b((X_n)_{n\geq 0}))$. ∎

We would like to define a measure-theoretic boundary to be a **Poisson boundary** if it solves the Dirichlet problem entirely. In view of that:

Theorem 6.1 (Poisson Boundary)**.** *Let $(B, \mathcal{F}_B, b)$ be a boundary of the irreducible Markov chain (V, P), equipped with the family of harmonic measures $(\nu_x)_{x\in V}$. The following are equivalent:*

(a) ***Poisson representation****: For all $u \in H^\infty(V, P)$, there exists a bounded measurable function $\tilde{u} : B \to \mathbb{R}$ such that:*

$$\forall x \in V, u(x) = \int_B \tilde{u}(\theta) d\nu_x(\theta)$$

(b) ***Harmonic limits/boundary convergence***: *For all* $u \in H^\infty(V,P)$ *there exists a bounded measurable function* $\tilde{u} : B \to \mathbb{R}$ *such that* $\lim_{n\to\infty} u(X_n) = \tilde{u}(b((X_n)_{n\geq 0}))$ *a.s.*
(c) ***Ergodic components***: $b^{-1}\mathcal{F}_B = \mathcal{I}(mod\ P_o)$.
(d) ***Maximality***: *If* $(B', \mathcal{F}'_B, b')$ *is a boundary of* (V,P), *then* $(B, \mathcal{F}_B, b)$ *factors through* $(B', \mathcal{F}'_B, b')$, *i.e. there exists a measurable map* $\pi : B \to B'$ *such that:* $b' = \pi \circ b$ P_o*-a.s.*

Proof. We refer the reader to [LP16] for a proof. ■

Remark. Observe that the Poisson boundary $\partial_\mu\Gamma$ defined for $RW_{\Gamma,\mu}$ satisfies (c) above.

6.3. **Boundary Convergence and Compactification Boundaries.**

The state space of our irreducible Markov chain, which is the *background* on which random movement occurs, might come equipped with natural topological/geometric structures possibly along with some group action, and thus some associated boundaries. On that note, suppose that V is a separable metric space.

Definition 6.2. A **compactification** of V is a compact, hausdorff, second countable metric space $\hat{V}$ together with an embedding $i : V \hookrightarrow \hat{V}$ such that $i(V)$ is open and dense in $\hat{V}$.

Remark. Observe that $\partial V = \hat{V} - i(v)$ is compact and that the metric on V extends continuously to $\hat{V}$.

- For any $\hat{x}, \hat{y} \in \hat{V}$ if $(x_n), (y_n)$ are any sequences in V that $(i(x_n)), (i(y_n))$ converge to $\hat{x}, \hat{y}$ respectively, then simply define $d(\hat{x}, \hat{y}) = \lim_{n\to\infty} d(x_n, y_n)$. Second countability of $\hat{V}$ and the fact that $i(V)$ is dense in $\hat{V}$ imply that $(x_n), (y_n)$ are Cauchy sequences in V. Existence of the limit now follows because d is uniformly continuous.

Let $\Omega_\infty = \{\omega = (x_n)_{n\geq 0} \in V^{\mathbb{N}} \mid Z_\infty(\omega) = \lim_{n\to\infty} Z_n(\omega) \in \partial V$ exists in the topology of $\hat{V}\}$. We say that there is **boundary convergence** if $\forall x \in V, P_x(\Omega_\infty) = 1$.

Remark. If there is boundary convergence, Z_∞ is measurable wrt to the Borel σ-algebra, $\mathcal{B}_{\partial V}$ of ∂V. Thus for each $x \in X$, we have the **hitting measures** $\nu_x = P_x \circ Z_\infty^{-1}$. Note that $(\nu_x)_{x\in V}$ is a family of harmonic measures. Together with irreducibility of the Markov chain, this implies that $(\nu_x)_{x\in V}$ are mutually absolutely continuous. Thus if $\varphi : \partial V \to \mathbb{R}$ is ν_x-integrable, then it is ν_y-integrable for all $y \in V$. As we saw in the previous section, this solves half of the Dirichlet problem: for all $\varphi \in C(\partial V)$, $\forall x \in V, u_\varphi(x) = \int_{\partial V} \varphi d\nu_x$ defines a harmonic function on V. But is this extension continuous?

Theorem 6.2. *The Dirichlet problem wrt to* P *and* $\hat{V}$ *is solvable iff:*
(a) ***Boundary convergence***: (Z_n) *converges to the boundary.*
(b) $\forall \xi \in \partial V$, $\lim_{x\to\xi} \nu_x = \delta_\xi$ *weakly.*

Proof. Observe that (a) and (b) clearly imply that the for all for all $\varphi \in C(\partial V)$, the harmonic function u_φ is a continuous extension of φ. So now, assume that the Dirichlet problem is solvable wrt to P and $\hat{V}$.

- For every $\varphi \in C(\partial V)$ let u_φ denote the (restriction to V of the) (unique) continuous harmonic extension to $\hat{V}$. Then the map defined by evaluation at $x \in V$, $\text{ev}_x : C(\partial V) \to \mathbb{R}$ defined by $\text{ev}_x(\varphi) = u_\varphi(x)$:
  - is linear.
  - $\text{ev}_x(1) = 1$ and for $\varphi \geq 0$, $\text{ev}_x(\varphi) \geq 0$.

  – $||\mathrm{ev}_x|| = 1$ (follows from the maximum principle)

  Thus by the Riesz-Markov theorem, there exists a unique Radon measure ν_x on ∂V such that $\mathrm{ev}_x(\varphi) = \int_{\partial V} \varphi d\nu_x$ for all $x \in V$, $\varphi \in C(\partial V)$.
- $\forall \varphi \in C(\partial V) \forall \xi \in \partial V$, $\lim_{x\to\xi} u_\varphi(x) = \varphi(\xi)$, i.e $\lim_{x\to\xi} \nu_x = \delta_\xi$. This proves (b).
- Next, we need to show that $\forall x \in V, P_x(\Omega_\infty) = 1$. As ∂V is compact, Hausdorff and second countable, $C(\partial V)$ (equipped with the sup norm) is separable. Let (φ_k) be a countable dense subset of $C(\partial V)$. For each k, the sequence $(u_{\varphi_k}(Z_n))$ is a bounded martingale, and so by the martingale convergence theorem, converges P_x-a.s. for all $x \in V$.
- Noting that the Markov chain is transient let

$$\Omega' = \{\omega = (x_n)_{n\geq 0} \in V^{\mathbb{N}} \mid Z_n(\omega) \to \infty \text{ and } (u_{\varphi_k}(Z_n(\omega))) \text{ converges}\}$$

  Then $P_x(\Omega') = 1$ for all $x \in V$.
- Let $\omega \in \Omega'$. By the Riesz-Markov theorem we get a unique Radon measure $\tilde{\nu}_\omega$ such that $\lim_{n\to\infty} \nu_{Z_n(\omega)} = \tilde{\nu}_\omega$. Since $\hat{V}$ is compact the sequence $Z_n(\omega)$ has at least one limit point. Since $Z_n(\omega) \to \infty$, all limit points belong to ∂V. If (n_i) is a subsequence such that $Z_{n_i}(\omega) \to \xi \in \partial V$ as $i \to \infty$, then (b) implies that $\lim_{i\to\infty} \nu_{Z_{n_i}(\omega)} = \delta_\xi = \tilde{\nu}_\omega$. In fact, this observation also shows that if η is any other limit points of $(Z_n(\omega))$, then $\delta_\xi = \tilde{\nu}_\omega = \delta_\eta$ i.e. $\xi = \eta$. This proves (a), since $\Omega' \subseteq \Omega_\infty$.
- It remains to check that for each $x \in V$, ν_x is the hitting measure for the Markov chain starting at x. It suffices to show that:

$$\forall x \in V \forall \varphi \in C(\partial V), E_x[\varphi \circ Z_\infty] = \int_{\partial V} \varphi d\nu_x$$

  Observe that $\varphi \circ Z_\infty = \lim_{n\to\infty} u_\varphi(Z_n)$ P_x-a.e. for all $x \in V$. Now by the dominated convergence theorem,

$$\begin{aligned} E_x[\varphi \circ Z_\infty] &= \lim_{n\to\infty} E_x[u_\varphi(x)] \\ &= \lim_{n\to\infty} P^n u_\varphi(x) \\ &= \lim_{n\to\infty} u_\varphi(x) = u_\varphi(x) \end{aligned}$$

  where $P^n u_\varphi(x) = u_\varphi(x)$ since u_φ is harmonic.

■

This leads us to define:

Definition 6.3. A measure-theoretic boundary $(B, \mathcal{F}_B, b)$ is called a **compactification boundary** if:

- There is a compactification $\hat{V}$ of V such that $B = \partial V$ and $\mathcal{F}_B = \mathcal{B}_{\partial V}$.
- (Z_n) converges to the boundary.
- $b\big|_{\Omega_\infty} = Z_\infty$ and on $V^{\mathbb{N}} - \Omega_\infty$, we simply require b to be T-invariant and measurable (like for example, it could take a constant value).

Now suppose Γ is a group of automorphisms of (V, P) which also acts on B such that b is Γ-equivariant (Here Γ acts diagonally on $V^{\mathbb{N}}$).

- $(B, \mathcal{F}_B, b)$ is said to be a **Γ-boundary** if Γ acts $\mathcal{F}_B$-measurably on B.
- If $(B, \mathcal{F}_B, b)$ is a compactification boundary and Γ acts continuously on B, the it is said to be a **compactification Γ-boundary**.

The main application of Theorem 6.2 lies in the fact that if we can show that the Dirichlet problem is solvable, then the 6.2 together with 6.1 realizes the Poisson boundary as a compactification boundary.

6.4. **Compactifications with hyperbolic properties.**

Let (X, d) be a proper, separable metric space, and Γ be a group of isometries of (X, d). Let $\hat{X}$ be a compactification of X.

Definition 6.4. $\hat{X}$ is said to be a **projective** compactification if:

- For all sequences $(x_n), (y_n)$ in X and for all $\xi \in \partial X$, whenever $x_n \to \xi$ and $\sup_n d(x_n, y_n) < \infty$ we have $y_n \to \xi$.

$\hat{X}$ is said to be a **contractive Γ-compactification** if:

- $\hat{X}$ is projective
- Every $\gamma \in \Gamma$ extends to a self homeomorphism of $\hat{X}$.
- For every sequence (γ_n) in Γ, for some $x \in X$ and for all $\xi, \eta \in \partial X$, if $\gamma_n x \to \xi$ and $\gamma_n^{-1} x \to \eta$ then $\gamma_n w \to \xi$ uniformly for all $w \in \hat{X}$ outside of every neighbourhood of η.

Remark.

- Note that since Γ acts by isometries, and due to projectivity, the convergence criterion for a contractive Γ-compactification does not depend on the choice of $x \in X$.
- The convergence criterion can be stated equivalently as follows: for any neigbourhoods U, V of ξ, η respectively, there is a non-negative integer $N = N(U, V)$ such that for all $n \geq N$, we have $\gamma_n(\hat{X} - V) \subseteq U$.

Theorem 6.3. *Let X be a countably infinite locally finite graph and consider an irreducible, transient Markov chain (X, P). Suppose Γ is a closed transitive subgroup of $Aut(X, P)$. If $\hat{X}$ is a contractive Γ-compactification of X such that:*

- $|\partial X| = \infty$
- Γ *does not fix any point on* ∂X

Then, the Dirichlet problem wrt to P and $\hat{X}$ is solvable.

We will now specialize to the case of random walks on countable groups.

6.5. **Proximal actions and μ-boundaries.**

In this section G will be a locally compact, second countable group, (M, d) will denote a compact metric space on which G acts by isometries, i.e. a compact metric G-space and $\mathrm{Prob}(M)$ will denote the space of Borel probability measures on M. $\delta_M := \{\delta_x \mid x \in M\}$

Definition 6.5.

- M is called a **minimal** G-space if it does not contain a proper closed G-invariant subspace.
- M is called a **proximal** G-space if for all $x, y \in M$, there is a sequence (g_n) in G such that $d(g_n x, g_n y) \to 0$. Since M is compact, we can equivalently say that for all $x, y \in M$ there is a point $z = z(x, y) \in M$ and a sequence (g_n) in G such that $\lim_{n\to\infty} g_n x = z = \lim_{n\to\infty} g_n y$.

- M is called a **strongly proximal** G-space if for all $\mu \in \text{Prob}(M)$, there exists a sequence (g_n) in G and a point $x \in M$ such that $g_n\mu \to \delta_x$.
- M is called a **boundary** for the group G if it is a minimal, strongly proximal G-space.
- Suppose $\mu \in \text{Prob}(G)$ and $\nu \in \text{Prob}(M)$ is a μ-stationary measure, i.e. $\mu * \nu = \nu$. Then (M, ν) is called a $(\mathbf{G}, \mu)$-space.
- A (G, μ)-space (M, ν) is called a μ**-boundary** if for all almost all sequences $(g_n)_{n\geq 1} \in G^{\mathbb{N}_{>0}}$, $\lim_{n\to\infty} g_1 \dots g_n\nu \in \delta_M$. Equivalently: If (X_n) is an i.i.d. sequence of G-valued random variables with law μ, then a.s. $e, X_1\nu, \dots, X_1 \dots X_n\nu$ converges to some point measure.
- Let $\mu \in \text{Prob}(G)$. M is called a μ**-proximal** G-space if for every μ-stationary measure $\nu \in \text{Prob}(M)$, the (G, μ)-space (M, ν) is a μ-boundary.
- M is called a **mean proximal** G-space if it is μ-proximal for every $\mu \in \text{Prob}(G)$ with $\text{supp}(\mu) = G$.

Proposition 6.3. *Let $\mu \in Prob(G)$ and let (M, ν) be a (G, μ)-space. Set $\Omega = G^{\mathbb{N}_{>0}}$ equipped with the product σ-algebra and product measure $P = \mu^{\otimes \mathbb{N}_{>0}}$. Then for P-almost every sequence $(g_n)_{n\geq 0} \in \Omega$, the limit:*

$$\lim_{n\to\infty} g_1 g_2 \dots g_n \nu \text{ exists.}$$

Proof. Let $X_n : \Omega \to G$ denote the projection onto n^{th} coordinate. Then $(X_n)_{n\geq 0}$ is an i.i.d. sequence of random variables with law μ. Define another G-valued sequence of random variables by: $Z_0 = e$ $Z_{n+1} = Z_n X_{n+1}$ for $n \geq 0$. The law of Z_n is $\mu^n := \mu^{*n}$. Consider any $f \in C(M)$. Consider the random variables: $Y_n = E_{Z_n\nu}[f]$. Note that $||Y_n||_1 \leq ||f||_{\text{sup}}$ and that (Y_n) is a martingale wrt the filtration $(\sigma(X_1, \dots, X_n))$. This follows from the following computation:

$$\begin{aligned} \mu * \nu = \nu \implies E_\nu[f] &= \int_G E_{g\nu}[f] d\mu(g) \\ \text{So } E_{Z_n\nu}[f] &= E_\nu[f \circ Z_n] \\ &= \int_G E_{g\nu}[f \circ Z_n] d\mu(g) \\ &= \int_G E_{Z_n g\nu} d\mu(g) \\ \text{i.e } Y_n &= E_P[Y_{n+1} \mid X_1, \dots, X_n] \end{aligned}$$

Thus, by the Martingale convergence theorem, Y_n converges P-a.s. Now note that M is a separable metric space, hence $C(M)$ is separable. Thus we may now conclude that P-a.s. the sequence $(Z_n\nu)$ converges. ■

6.6. **Kaimanovich's Strip Convergence.**

Let G be a countable group. Following Kaimanovich, we consider a compactification $\overline{G} = G \cup \partial G$ of G satisfying the following conditions:

- **CE:** The left G-action of G on itself, extends to an action of G on $\overline{G}$ by homeomorphisms.
- **CP:** For any sequence (g_n) in G, for any $\xi \in \partial G$, if $g_n \to \xi$ in $\overline{G}$, then for all $x \in G$, $g_n x \to \xi$.

- **CS:** $|\partial G| \geq 3$ and there is a G-equivariant Borel map $S : \partial_\infty^{(2)} G \to \mathcal{P}(G)$: for every distinct pair $\xi_1, \xi_2 \in \partial G$, we call $S(\xi_1, \xi_2)$ a **strip**. S satisfies the following:
  - For any distinct points $\bar{\xi}_0, \bar{\xi}_1, \bar{\xi}_2 \in \partial G$, there exist neighbourhoods: $\mathcal{O}_0 \subset_{\text{open}} \overline{G}$ containing $\bar{\xi}_0$, $\mathcal{O}_i \subset_{\text{open}} \partial G$ containing $\bar{\xi}_i$ for $i = 1, 2$ such that for all $\xi_i \in O_i$, we have $S(\xi_1, \xi_2) \cap \mathcal{O}_0 = \phi$.

Lemma 6.1. *Suppose $\overline{G} = G \cup \partial G$ is a compactification of G satisfying (CE), (CP) and (CS). Let $\xi \in \partial G$ and (g_n) be a sequence in G converging to ξ. For any non-atomic Borel probability measure λ on ∂G, $g_n \lambda$ converges to δ_ξ weakly.*

Proof. To begin with, if for all $b \in \partial G$, $g_n b \to \xi$, then we're done. This is because for any $\varphi \in C(\partial G)$, $E_{g_n \lambda}[\varphi] = E_\lambda[\varphi \circ g_n] \to E_\lambda[\varphi(\xi)] = \varphi(\xi)$. So suppose there is point $b_1 \in \partial G$ and g_n converges to $\xi' \neq \xi$ (after passing to a subsequence if necessary). We claim that for all $b \in \partial G - \{b_1\}$, $g_n b$ converges to ξ. If not, there exists a point $b_2 \in \partial G - \{b_1\}$ such that $g_n b_2$ converges to $\xi'' \neq \xi$ (again after passing to a subsequence if necessary). Now choose any $x \in S(b_1, b_2)$. By (CP), $g_n x$ converges to ξ. But by G-equivariance, $g_n x \in S(g_n b_1, g_n b_2)$, and so $\xi \in S(\xi', \xi'')$ which contradicts (CS). Since λ is non-atomic, it follows that $g_n \lambda$ converges to δ_ξ. ■

Remark. In fact, this shows that ∂G is a strongly proximal G-space.

Theorem 6.4. *Suppose $\overline{G} = G \cup \partial G$ is a compactification of G satisfying (CE), (CP) and (CS) and μ is a probability measure on G such that the subgroup generated by supp(μ) is non-elementary wrt to the compactification. Then:*

1. *Then P_e-a.e. sample path $\omega = (x_n)$ converges to a limit $Z_\infty(\omega) \in \partial G$.*
2. *The limit measure $\lambda = (Z_\infty)_* P_e$ is purely non-atomic.*
3. *$(\partial G, \lambda)$ is a μ-boundary for (G, μ) and λ is the unique μ-stationary measure on ∂G.*

Proof. Since ∂G is a compact, separable metric space, so is Prob(∂G). By Schauder's fixed point theorem, there exists a μ-stationary measure ν on ∂G. Before we begin, maybe it useful to recall that:

- *step space*= $\Omega = G^{\mathbb{N}_{>0}}$ equipped with the product σ-algebra and the product probability measure $\mu^{\mathbb{N}_{>0}}$. If X_n denotes the projection of a *step sequence* onto the n^{th} coordinate, then the random walk is given by the random variables: $Z_1 = e, Z_{n+1} = Z_n X_n$ for $n \geq 1$.
- *path space*= Ω equipped with the product σ-algebra and P_e, the pushforward of $\mu^{\mathbb{N}_{>0}}$ under the map $\Omega \to \Omega$ which maps each $\omega \in \Omega$ to $(Z_n(\omega))$.
- sgr(μ) refers to the semi-group generated by the support of μ while gr(μ) referes to the group generated by the support of μ.

Now we proceed with the proof:

- ν is non-atomic: Suppose on the contrary, that ν has atoms and let A_m be the (necessarily finite) set of atoms of maximal weight. For any $b \in A_m$, by stationarity we have:
$$\nu(b) = \sum_{g \in G} \mu(g) \nu(g^{-1} b) \quad (*)$$
Because the set of probabilities associated to the atoms is a discrete set and m is maximal among them, it follows that $\nu(g^{-1}b) \in A_m$ for all $g \in \text{supp}(\mu)$. Choice of $b \in A_m$ was arbitrary, so for all $g \in \text{supp}(\mu)$, $g^{-1} A_m \subseteq A_m$ and as A_m is finite,

$g^{-1}A_m = A_m$ i.e. $gA_m = A_m$, so A_m is fixed by $\mathrm{gr}(\mu)$. This is not possible though, because $\mathrm{gr}(\mu)$ is non-elementary and hence by definition, cannot fix any subset of ∂G.

- Boundary convergence: We emphasize here that one needs $\mathbf{sgr}(\mu)$ to be non-elementary here in order to conclude that the random walk is transient, so that P_e-almost every random sample path ω escapes to infinity, i.e. converges to a point say $Z_\infty(\omega)$ in ∂G. Note that Z_∞ is measurable wrt the Borel σ-algebra of ∂B.
- $(\partial G, \nu)$ is a μ-boundary for (G, μ): ν is μ-stationary, so it follows from proposition 6.3 that for P-a.e. *step* sequence $(\mathbf{g_n})$ in G, $(Z_k((\mathbf{g_n}))\nu)$ converges to a probability measure $\nu_{(\mathbf{g_n})}$. In other words, for P_e-almost every *sample path* $\omega = (x_n)$, the sequence of probabilities $(x_n\nu)$ converges to a probability measure ν_ω. Now we observe that lemma 6.1 implies $\nu_\omega = \delta_{Z_\infty(\omega)}$.
- $\nu = \lambda$: Consider a random sample path (x_n). The stationarity of ν applied n times gives:

$$\begin{aligned}\forall A \in \mathcal{B}_{\partial G} \quad \nu(A) &= \mu^n * \nu(A) \\ &= \sum_g \mu^n(g)(g\nu)(A) \\ &= E_{P_e}[(Z_n\nu)(A)] \\ &= E_{P_e}[\delta_{Z_\infty}(A)] = P_e(Z_\infty \in A) = \lambda(A)\end{aligned}$$

where in the last step we have used the fact that $Z_n\nu$ converges to δ_{Z_∞} weakly as $n \to \infty$.

■

Remark. It follows that $(\partial G, \lambda)$ is a mean-proximal G-space.

7. The entropy criterion

In this section we will give an entropy criterion for determining when a μ-boundary is maximal, that is, a Poisson boundary. Let us briefly recall some notation.

- Let G be a countable group equipped with a probability measure μ, this is the *law of the steps* taken by the random walk. Call the product space $\Omega = G^{\mathbb{N}}$ as the **path space**: an element of Ω is to be thought of as a sample path of the random walk. Call the product space $\Omega_{\text{step}} = G^{\mathbb{N}_{>0}}$ as the **step space**: an element of Ω_{step} is to be thought of as the full sequence of *steps* taken by the random walk.
- We have the *walk* map, $\mathrm{walk} : G \times \Omega_{step} \to \Omega$ defined by $\mathrm{walk}(x_0, (x_n)_{n\geq 1}) = (w_n)_{n\geq 0}$ where $w_n = x_0 \dots x_n$. Equip Ω_{step} with the product probability measure $\mu^{\otimes \mathbb{N}_{>0}}$. Given an *initial distribution* θ on G, the law of the random walk is given by $P_\theta = \mathrm{walk}_*(\theta \otimes \mu^{\otimes \mathbb{N}_{>0}})$. We set $P = P_{\delta_e}$, the law of the random walk starting at identity.
- Note that G acts on sample paths coordinate-wise and that the this action commutes with the **time shift** map $T : \Omega \to \Omega$. The **Poisson boundary** of G is the space of ergodic components of (Ω, P, T) and we will denote it by $\partial_\mu G$ and it comes with the projection map $\mathrm{bnd} : \Omega \to \partial_\mu G$. Since the G-action on Ω commutes with T, we get a G-action on $\partial_\mu G$. $\nu = \mathrm{bnd}_* P$ is called the **harmonic measure**. Note that bnd is T-invariant and therefore ν is μ-stationary.
- The standard Bernoulli shift U_{step} on Ω_{step} (mapping (x_n) to (x_{n+1})) induces a Bernoulli shift U on Ω as given below:

$$\begin{array}{ccc} G\times\Omega_{\text{step}} & \xrightarrow{id_G\times U_{\text{step}}} & G\times\Omega_{\text{step}} \\ \big\downarrow{\scriptstyle\text{walk}} & & \big\downarrow{\scriptstyle\text{walk}} \\ \Omega & \xrightarrow{\quad U\quad} & \Omega \end{array}$$

Also observe that $P = \text{walk}^e_*(\mu^{\mathbb{N}_{>0}})$. U_{step} is measure-preserving and ergodic, therefore so is U. T, bnd, U are related as follows:

Lemma 7.1. *For P-a.e. sample path $\boldsymbol{w} = (w_n)$ in Ω, we have: $T(\boldsymbol{w}) = w_1 U(\boldsymbol{w})$ and $bnd(\boldsymbol{w}) = w_1 bnd(U\boldsymbol{w})$.*

Proof. If $\mathbf{w} = (w_n)_{b\geq 0}$ is a sample path starting at identity then $U(\mathbf{w})_0 = e$ and $U(\mathbf{w})_n = w_1^{-1}w_{n+1}$, so it is clear that $T(\mathbf{w}) = w_1 U(\mathbf{w})$. The second identity follows from this using G-equivariance and T-invariance of bnd. Note that P is supported on the set of sample paths starting at identity. This completes the proof. ∎

- A μ-**boundary**, B, is a quotient of the Poisson boundary with by a G-equivariant measurable partition. Denote by bnd_B the composition of bnd with the projection of $\partial_\mu G$ onto B. Set $\lambda = (\text{bnd}_B)_* P$. λ is the push-forward of the harmonic measure under $\partial_\mu G \twoheadrightarrow B$.
- We will also set up some notation for cylinder sets:

$$C^n_g = \{\mathbf{w}\in\Omega \mid w_n = g\}, C^{n_1,\ldots,n_k}_{g_1,\ldots,g_k} = \cap_{i=1}^k C^{n_i}_{g_i}, C_{g_1,\ldots,g_k} = \cap_{i=0}^k C^i_{g_i}$$

Now suppose we have a μ-boundary (B,λ). Our next goal is to compute/understand the conditional probabilities: $\forall\gamma\in B$, $P^\gamma := P(\cdot \mid \text{bnd}_B = \gamma)$.

Let A be a measurable subset of the μ-boundary B such that $\lambda(A) > 0$. The conditional probability measure $P^A := P(\cdot \mid \text{bnd}_B \in A)$ is uniquely determined by the values it takes on cylinder sets $C_{e,g_1,\ldots,g_k}$ where $k\geq 0$ and $g_i\in G$ for $i\leq k$.

$$\begin{aligned} &P(C_{e,g_1,\ldots,g_k}\cap[\text{bnd}_B(\mathbf{w})\in A]) \\ &= P(w_0 = e, w_1 = g_1,\ldots,w_k = g_k \cap [\text{bnd}_B(\mathbf{w})\in A]) \\ &= P(w_0 = e, w_1 = g_1,\ldots,w_k = g_k \cap [\text{bnd}_B\circ T^k(\mathbf{w})\in A]) \\ &= P[w_0 = e, w_1 = g_1,\ldots,w_k = g_k] P[\text{bnd}_B\circ T^k(\mathbf{w})\in A \mid w_0 = e, w_1 = g_1,\ldots,w_k = g_k] \\ &= P[w_0 = e, w_1 = g_1,\ldots,w_k = g_k] P_{g_k}[\text{bnd}_B(\mathbf{w})\in A] \\ &= P[w_0 = e, w_1 = g_1,\ldots,w_k = g_k] g_k\lambda(A) \end{aligned}$$

where in the 4th step we have used the Markov property. Dividing both sides by $\lambda(A)$ gives:

$$\begin{aligned} P(w_0 = e, w_1 = g_1,\ldots,w_k = g_k \mid \text{bnd}_B(\mathbf{w})\in A) &= P[w_0 = e, w_1 = g_1,\ldots,w_k = g_k]\frac{g_k\lambda(A)}{\lambda(A)} \\ &= P[w_0 = e, w_1 = g_1,\ldots,w_k = g_k]\varphi_A(g_k) \end{aligned}$$

where φ^A is the μ-harmonic function on G defined by

$$\varphi_A(x) = \frac{x\lambda(A)}{\lambda(A)} = \frac{1}{\lambda(A)}\int_A \frac{dx\lambda}{d\lambda} d\lambda$$

Definition 7.1 (Doob transform)**.** Let $H_1^+(G,\mu)$ denote the set of non-negative harmonic functions f defined on $\mathrm{sgr}(\mu)$ such that $f(e)=1$. Given $f \in H_1^+(G,\mu)$, we can construct a new Markov chain on $\mathrm{sgr}(\mu)$ with transition probabilities:

$$\forall x, y \in G, p^f(x,y) = \mu(x^{-1}y)\frac{f(y)}{f(x)}$$

Denote by P^f, the law of the new markov chain starting at the identity.

Remark. For $f \in H_1^+(G,\mu)$, observe that $P^f(w_0 = e, w_1 = g_1, \dots, w_k = g_k) = P(w_0 = e, w_1 = g_1, \dots, w_k = g_k)f(g_k)$, thus the map $f \mapsto P^f$ is affine.

Continuing our computation, define φ_γ to be the harmonic function on G defined by $\varphi_\gamma(x) = \frac{dx\lambda}{d\lambda}(\gamma)$. Then:

- $\varphi_A \in H_1^+(G,\mu)$, $\varphi_A(\cdot) = 1/\lambda(A)\int_A \varphi_\gamma(\cdot)d\lambda(\gamma)$.
- $P(\cdot \mid \mathrm{bnd}_B \in A)$ is the Doob transform of P wrt φ_A and:

$$P(\cdot \mid \mathrm{bnd}_B \in A) = \frac{1}{\lambda(A)}\int_A P^\gamma(\cdot)d\lambda(\gamma)$$

which shows in particular, in view of Rokhlin's theorem on disintegration of measures, that:

Theorem 7.1. *The probability measures P^γ, corresponding to the transition probabilities $p^\gamma(x,y) = \mu(x^{-1}y)dy\lambda/dx\lambda(\gamma)$ are the conditional probabilities constituting a disintegration of (Ω, P) wrt to the partition determined by the μ-boundary: $(bnd_B^{-1}(\gamma))_{\gamma\in B}$.*

We now recall the notions of **conditional** and **asymptotic** entropy adapted to our setup. Suppose X is a discrete random variable defined on the path space and $\mathcal{F}$ is a σ-algbera (consisting of some Borel sets). The conditional entropy of X wrt $\mathcal{F}$ is defined as:

$$H(X \mid \mathcal{F}) = E[-\sum_x P(X = x \mid \mathcal{F}) \log P(X = x \mid \mathcal{F})]$$

We will especially be interesting in the case when $\mathcal{F} = \sigma(\mathrm{bnd})$ or $\mathcal{F} = \sigma(\mathrm{bnd}_B)$. A probability measure Λ on the path space is said to have asymptotic entropy $h(\Lambda)$ if it satisfied the following asymptotic equipartition property:

$$-\frac{1}{n}\log \Lambda \circ W_n \to h(\Lambda) \quad \Lambda\text{-a.s. and in } L^1(\Lambda)$$

where W_n is the random variable denoting the location of the random walk at time n.

Lemma 7.2. *Let (B, λ, bnd_B) be a μ-boundary of G and $H(\mu) < \infty$. For all $k \geq 1$,*

$$\begin{aligned} H((W_i)_{i\leq k} \mid bnd_B) &= kH(W_1 \mid bnd_B) \\ &= k\left[H(\mu) - \int \log \frac{dw_1\lambda}{d\lambda}(bnd_B\boldsymbol{w})dP(bnd_B\boldsymbol{w})\right] \end{aligned}$$

Proof. Observe first that for all $\mathbf{w} = (w_n)_{n\geq 0}$,

$$\begin{aligned} P(W_0 = e, W_1 = w_1, \dots, W_k = w_k \mid \mathrm{bnd}_B) &= P(W_0 = e, W_1 = w_1, \dots, W_k = w_k \mid \mathrm{bnd}_B\mathbf{w}) \\ &= P^{\mathrm{bnd}_B\mathbf{w}}(W_0 = e, W_1 = w_1, \dots, W_k = w_k) \\ &= P(W_0 = e, W_1 = w_1, \dots, W_k = w_k)\frac{dw_k\lambda}{d\lambda}(\mathrm{bnd}_B\mathbf{w}) \end{aligned}$$

We will be using a clever formula (possibly due Rokhlin):

$$H((W_i)_{i\le k} \mid \mathrm{bnd}_B) = -\int \log P(\mathbf{w}, (W_i)_{i\le k} \mid \mathrm{bnd}_B) dP(\mathbf{w})$$

which gives us:

$$H((W_i)_{i\le k} \mid \mathrm{bnd}_B) = -H(\mu^k) - \int \log \frac{dw_k\lambda}{d\lambda}(\mathrm{bnd}_B\mathbf{w})dP(\mathbf{w})$$

We use the cocycle property of Radon-Nikodym derivatives, to transform the integrand in the second term into a product:

$$\begin{aligned}\frac{dw_k\lambda}{d\lambda}(\mathrm{bnd}_B\mathbf{w}) &= \prod_{i=1}^{n} \frac{dx_1\dots x_i\lambda}{dx_1\dots x_{i-1}\lambda}(\mathrm{bnd}_B\mathbf{w})\\ &= \prod_{i=1}^{n} \frac{dx_i\lambda}{d\lambda}(w_{i-1}^{-1}\mathrm{bnd}_B\mathbf{w})\\ &= \prod_{i=1}^{n} \frac{d(U^{i-1}\mathbf{w})_1\lambda}{d\lambda}(\mathrm{bnd}_B(U^{i-1}\mathbf{w})) \quad (7.1)\end{aligned}$$

Now writing the log of the product into sum of logs and using change of variables for powers of U, noting that U is a measure-preserving map completes the proof. Note that $H(\mu^k) = kH(\mu)$ ■

Lemma 7.3. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two G-equivariant measurable partitions of $(\partial_\mu G, \nu)$ such that $\mathcal{Q}$ is finer than $\mathcal{P}$. Let $bnd_{\mathcal{P}}$ and $bnd_{\mathcal{Q}}$ denote the resepective μ-boundary maps. Then $H(W_1 \mid bnd_{\mathcal{P}}) \ge H(W_1 \mid bnd_{\mathcal{Q}})$ and equality holds iff $\mathcal{P} = \mathcal{Q}$.*

Lemma 7.4. *Suppose $H(\mu) < \infty$. We have $H(W_1 \mid \mathcal{T}) = H(\mu) - h(G, \mu)$, where $h(G, \mu)$ is the asymptotic entropy of the random walk. Here $\mathcal{T}$ is the tail σ-algebra.*

Proof. Finiteness of $H(\mu)$ implies that $h(G, \mu)$ exists and in fact:

$$-\frac{1}{n}\log \mu^n \circ W_n \to h(G, \mu) \quad \text{a.s. and in } L^1(P)\dots(*)$$

Let $n > 1$ then for any $k < N$ we have: $H(W_k \mid W_n) + H(W_n) = H(W_k, W_n) = H(W_n \mid W_k) + H(W_k)$ But $H(W_n \mid W_k) = H(W_{n-k})$ by Markov property, so $H(W_k \mid W_n) = H(W_k) + H(W_{n-k}) - H(W_n)$. Again, by Markov property $H(W_k \mid (W_j)_{j\ge n}) = H(W_k \mid W_n)$. Thus we have:

$$H(W_k \mid (W_j)_{j\ge n}) = H(W_k) + H(W_{n-k}) - H(W_n)$$

Now taking limit $n \to \infty$: $(*)$ implies that $H(W_{n-k}) - H(W_n) = \sum_{i=1}^{k} H(W_{n-i}) - H(W_{n-i+1}) \to -kh(G, \mu)$ while by monotonicity of conditional entropy, $H(W_k \mid (W_j)_{j\ge n}) \to H(W_k \mid \mathcal{T})$ where $\mathcal{T}$ is the tail σ-algebra. So:

$$H(W_k \mid \mathcal{T}) = k(H(\mu) - h(G, \mu))$$

■

Theorem 7.2. *Let (B, λ, bnd_B) be a μ-boundary of G. Then for λ-a.e. $\gamma \in B$, we have:*

$$h(P^\gamma) = H(W_1 \mid bnd_B) - H(W_1 \mid bnd)$$

Proof. By definition, we need to show that for λ-a.e., for P^γ-a.e. $\mathbf{w} = (w_n) \in \Omega$, $\gamma \in B$ we have:

$$-\frac{1}{n} \log P^\gamma([W_n = w_n]) \to H(W_1 \mid \text{bnd}_B) - H(W_1 \mid \text{bnd}) \text{ and also in } L^1(P^\gamma)$$

By construction, the conditional probabilities P^γ constitute a disintegration of P wrt to the measurable partition induced by bnd_B, $P(\cdot) = \int P^\gamma(\cdot) d\lambda(\gamma)$. Thus it suffices to show that for P-a.e. $\mathbf{w} \in \Omega$, we have:

$$-\frac{1}{n} \log P^{\text{bnd}_B \mathbf{w}}([W_n = w_n]) \to H(W_1 \mid \text{bnd}_B) - H(W_1 \mid \text{bnd}) \text{ and also in } L^1(P^)$$

Note that $P^\gamma([W_n = w_n]) = P([W_n = w_n]) dw_n\lambda/d\lambda(\gamma)$, so:

$$\begin{aligned}
\frac{\log P^{\text{bnd}_B \mathbf{w}}([W_n = w_n])}{n} &= \frac{\log P([W_n = w_n])}{n} + \frac{1}{n} \log \frac{dw_n\lambda}{d\lambda}(\text{bnd}_B \mathbf{w}) \\
&= \frac{\log P([W_n = w_n])}{n} + \frac{1}{n} \sum_{i=1}^{n} \frac{d(U^{i-1}\mathbf{w})_1\lambda}{d\lambda}(\text{bnd}_B(U^{i-1}\mathbf{w})) \\
&= \frac{P(\log[W_n = w_n])}{n} + \frac{1}{n} \sum_{i=1}^{n} f(U^{i-1}\mathbf{w})
\end{aligned}$$

where $f : \Omega \to \mathbb{R}$ is the function $f(\mathbf{w}) = \log \frac{dw_1\lambda}{d\lambda}(\text{bnd}_B \mathbf{w})$. Observe that f is a P-integrable function due to lemma 7.2. Since (Ω, P, U) is an ergodic system, by Birkhoff's ergodic theorem,

$$\frac{1}{n} \sum_{i=1}^{n} f(U^{i-1}\mathbf{w}) \to \int_\Omega f dP = H(\mu) - H(W_1 \mid \text{bnd}_B) \text{ a.s. and in } L^1$$

while by Kingman's subadditive ergodic theorem and the fact that $(\Omega_{step}, \mu^{\otimes \mathbb{N}_{>0}}, U_{\text{step}})$ is an ergodic system implies,

$$\frac{\log P([W_n = w_n])}{n} = \frac{\log \mu^n(w_n)}{n} \to -h(G, \mu) \text{ a.s. and in } L^1$$

Now we simply use 7.4 to complete the proof.

Remark. $\sigma(\text{bnd}) = \mathcal{I}(\text{mod } P)$ and we have used the fact that in this setting $\mathcal{I} = \mathcal{T}(\text{mod } P)$.

■

Now theorem 7.2 and lemma 7.3 give the promised entropy criterion

Theorem 7.3 (Entropy Criterion)**.** *A μ-boundary (B, λ) is the Poisson boundary iff the **asymptotic conditional entropies** $h(P^\gamma) = 0$ for λ-a.e. $\gamma \in B$.*

We derive a corollary which characterizes vanishing of the asymptotic conditional entropies in terms of growth rate of typical sets visited by the conditional random walk (conditioned to hit some point on a μ-boundary).

Corollary 7.3.1 (Kaimanovich's Enumeration Criterion)**.** *A μ-boundary (B, λ) of (G, μ) is the Poisson boundary if and only if for λ-a.e. point $\gamma \in B$, there exists an $\varepsilon > 0$ and a sequence of subsets of G, $(A_n = A_n(\gamma))_{n \geq 1}$ such that:*

(1) $\log |A_n| = o(n)$

(2) *For sufficiently large* n*,* $P_n^\gamma(A_n) > \varepsilon$ *where* $P_n^\gamma := (W_n)_* P^\gamma$ *is law of the* n^{th} *step of the random walk conditioned to "hit"* γ*.*

Proof. $\Longleftarrow$: By the entropy criterion (theorem 7.3) it suffices to show that $h(P^\gamma) = 0$ for λ-a.e. point $\gamma \in B$. Choose any point $\gamma \in B$ for which there exists an $\varepsilon > 0$ and a sequence of sets $(A_n)_{n\geq 1}$ and conditions $(1), (2)$ hold. Suppose that $h(P^\gamma) > 0$. Note that:

$$\frac{-1}{n} \log P_n^\gamma \circ W_n \to h(P^\gamma) \quad P^\gamma \text{ a.s. and in } L^1(P^\gamma) \quad (*)$$

Define the *typical* sets $S_n = \{g \in G \mid P_n^\gamma(g) \leq e^{-nh(P^\gamma)/2}\}$. Almost sure convergence in $(*)$ implies that $\lim_{n\to\infty} P^\gamma[W_n \notin S_n] = 0$. Then:

$$\begin{aligned} P^\gamma(W_n \in A_n \cap S_n) = \sum_{g \in A_n \cap S_n} P^\gamma(W_n = g) = \sum_{g\in A_n\cap S_n} P_n^\gamma(g) &\leq |A_n \cap S_n| e^{-nh(P^\gamma)/2} \\ &\leq |A_n| e^{-nh(P^\gamma)/2} \end{aligned}$$

So that (1) now gives: $\lim_{n\to\infty} P^\gamma(W_n \in A_n \cap S_n) = 0$. But:

$$\begin{aligned} P^\gamma(W_n \in A_n) &= P^\gamma(W_n \in A_n \cap S_n) + P^\gamma(W_n \in A_n \cap S_n^c) \\ &\leq P^\gamma(W_n \in A_n \cap S_n) + P^\gamma(W_n \notin S_n) \end{aligned}$$

$$\text{So } \lim_{n\to\infty} P^\gamma(W_n \in A_n) = 0$$

which contradicts (2). Thus $h(P^\gamma) = 0$.

$\Longrightarrow$: By the entropy criterion, we know that for λ-a.e. $\gamma \in B$ we have $h(P^\gamma) = 0$. Consider any such $\gamma \in B$. Fix $\varepsilon > 0$. Define $B_n = \{g \in G \mid P_n^\gamma(g) \geq 1/n^\varepsilon\}$. Then almost sure convergence in $(*)$ implies that $\lim_{n\to\infty} P^\gamma(W_n \in B_n) = 1$. So for sufficiently large n, $P^\gamma(W_n \in B_n) = P_n^\gamma(B_n) > \varepsilon$. Also, $|B_n| \leq n^\varepsilon$, so $\log |B_n| = o(n)$. ■

8. Ray approximation

Given a countable group G we define a **gauge** on G to be sequence of exhausting sets $\mathcal{G} = (G_k)_{k\geq 1}$. A gauge induces (and is uniquely determined by) the **gauge function** $|\cdot|_\mathcal{G} : G \to \mathbb{N}_{>0}$ defined by $|x|_\mathcal{G} = \min\{k \in \mathbb{N}_{>0} \mid x \in G_k\}$. We will say that a gauge $\mathcal{G} = (G_k)_{k\geq 1}$ is:

- **symmetric**: if for all k, $G_k = G_k^{-1}$ or equivalently $|\cdot|_\mathcal{G}$ is symmetric.
- **subadditive**: if for all k, l, $G_k G_l \subset G_{k+l}$ or equivalently, for all $x, y \in G$, $|xy|_\mathcal{G} \leq |x|_\mathcal{G} + |y|_\mathcal{G}$.
- **finite**: if for all k, G_k is finite or equivalently, all fibers of $|\cdot|_\mathcal{G}$ are finite.
- **temperate**: if $\sup_k \frac{1}{k} \log |G_k|$ is finite.

A family of gauges $(\mathcal{G}_\alpha)$ will be called **uniformly temperate** if $\sup_{k,\alpha} \frac{1}{k} \log |G_k|$ is finite.

Remark. Note that family of G-translates $(\mathcal{G}_g)_{g\in G}$ of a temperate gauge $\mathcal{G}$: $\mathcal{G}_g = g\mathcal{G} = (gG_k)_{k\geq 1}$ is uniformly temperate.

Let us now consider a natural class of gauges on a countable group G: namely word gauges. $\mathcal{G}$ is said to be a word gauge if G_1 generates G as a semigroup and $G_k = G_1^k$. Observe that:

- $\mathcal{G}$ is finite (resp. symmetric) iff G_1 is finite (resp. symmetric).
- $|\cdot|_\mathcal{G}$ coincided with the word length wrt to the generating set G_1 and G_k is the set of groups elements with word length $\leq k$.

- A finite word gauge is also temperate.
- If $\mathcal{G}, \mathcal{G}'$ are any two finite word gauges then $\forall g \in G$, $C^{-1}|g|_{\mathcal{G}} \leq |g|_{\mathcal{G}'} \leq C|g|_{\mathcal{G}}$. For example, $C = \max\{|g|_{\mathcal{G}'}, |g'|_{\mathcal{G}} \mid g \in G_1, g' \in G'_1\}$ works.

Given a measured group (G, μ) and a μ-boundary $(B, \lambda, \text{bnd}_B)$, we would like to assign a *ray* to every boundary point $\gamma \in B$. One way of doing this in a reasonable manner is the following:

- Construct measurable maps $\pi_n : B \to G$. Then $(\pi_n(\gamma))_{n\geq 1}$ could be thought of as an *abstract ray* travelling to γ.

Now suppose we are given a decent gauge $\mathcal{G}$ on G. Consider the family of G-translates $(\mathcal{G}_g)_{g\in G}$ of this gauge. Then $|h|_{\mathcal{G}_g} = |g^{-1}h|_{\mathcal{G}}$. The function $d : G \times G \to \mathbb{N}_{>0}$ defined by $d(g, h) = |h|_{\mathcal{G}_g}$ for $g, h \in G$ gives a measure of separation between g, h: distance of h from g. In fact, when $\mathcal{G}$ is symmetric and subadditive, d is a genuine metric. Thus it is reasonable to say thatthe rays given by (π_n) **approximate sample paths** if they track sample paths sublinearly:

- $d(\pi_n(\text{bnd}_B(\mathbf{w})), w_n) = o(n)$ for P-a.e. sample path $\mathbf{w} = (w_n) \in \Omega$.

We also expect the tracking asymptotics to somehow reflect the "geometry" of the group as determined by d. We will see in theorem 8.2 below that ray approximation together with subexponential growth of gauge sets guarantees that the μ-boundary under consideration is the Poisson boundary of (G, μ). Before that, we prove a similar result using a more synthetic ray approximation below.

Theorem 8.1. *Let (G, μ) be a measured group with finite entropy i.e. $H(\mu) < \infty$ and let (B, λ, bnd_B) be a μ-boundary. If for λ-a.e. point $\gamma \in B$, there exists a uniformly temperate sequence of gauges $(\mathcal{G}_n)_{n\geq 1} = (\mathcal{G}_n(\gamma))_{n\geq 1}$ where $\mathcal{G}_n(\gamma) = (G_{n,k}(\gamma))_{k\geq 1}$ such that:*

$$\lim_{n\to\infty} \frac{1}{n}|w_n|_{\mathcal{G}_n(bnd_B(\boldsymbol{w}))} = 0 \quad \textit{for } P\textit{-a.e. sample path } \boldsymbol{w} \in \Omega \quad (*)$$

then (B, λ) is the Poisson boundary of (G, μ).

Proof. $(*)$ implies that for λ-a.e. $\gamma \in B$, and for P^γ-a.e. $\mathbf{w} \in \Omega$, we have

$$\lim_{n\to\infty} \frac{1}{n}|w_n|_{\mathcal{G}_n(\gamma)} = 0$$

Consider any such γ and the sets $A_n = \{g \in G \mid |g|_{\mathcal{G}_n(\gamma)} \leq \sqrt{n}\}$. Then $|A_n| = \sum_{k\leq\lfloor\sqrt{n}\rfloor} G_{n,k}(\gamma) \leq C\sqrt{n}$ for some constant C (since $(\mathcal{G}_n(\gamma))_{n\geq 1}$ is uniformly temperate). So:

- $\log|A_n| = o(n)$
- $\lim_{n\to\infty} P^\gamma(W_n \in A_n) = 1$

So corollory 7.3.1 implies that (B, λ) is the Poisson boundary of (G, μ). ■

Theorem 8.2 (Ray approximation)**.** *Let (G, μ) be a measured group with finite entropy i.e. $H(\mu) < \infty$ and let (B, λ, bnd_B) be a μ-boundary. If there exists a temperate gauge $\mathcal{G}$ and a sequence of measurable maps $(\pi_n : B \to G)_{n\geq 1}$ such that:*

$$\lim_{n\to\infty} \frac{1}{n}|\pi_n(bnd_B(\boldsymbol{w}))^{-1}w_n|_{\mathcal{G}} = 0 \quad \textit{for } P\textit{-a.e sample path } \boldsymbol{w} \in \Omega$$

then (B, λ) is the Poisson boundary of (G, μ).

Proof. Follows from theorem 8.1 using the gauges $(\mathcal{G}_n(\gamma))_{n\geq 1,\gamma\in B}$ defined by:

$$\mathcal{G}_n(\gamma) = (G_{n,k}(\gamma))_{k\geq 1}, G_{n,k}(\gamma) = \pi_n(\gamma)\mathcal{G}$$

■

9. Strip approximation

We begin by introducing the space of **bilateral paths** $(\overline{\Omega}, \overline{P})$.

- $\overline{\Omega} = G^{\mathbb{Z}}$ equipped with the product σ-algebra.
- Until now we have been looking at the (unilateral) path space (Ω, P) where $\Omega = G^{\mathbb{N}}$, $P = p_*\mu^{\otimes \mathbb{N}_{>0}}$ and the map $p : \Omega_{\text{step}}(= G^{\mathbb{N}_{>0}}) \to \Omega$ is defined by $p((x_n)_{n\geq 1}) = (w_n)_{n\geq 0}$:
$$w_0 = e, \quad w_n = w_{n-1}x_n \text{ for } n \geq 1 \quad *$$
- If we are given a full sequence $(x_n)_{n\in\mathbb{Z}}$, then using the same recursive formula in $(*)$, we can extend $(w_n)_{n\geq 0}$ to a full sequence defined over $\mathbb{Z}$. More precisely, we define a map $q : \overline{\Omega}_{\text{step}} \to \overline{\Omega}$ where $\overline{\Omega}_{\text{step}} = G^{\mathbb{Z}}$ equipped with the product σ-algebra and $\mu^{\otimes\mathbb{Z}}$, by $q((x_n)) = (\overline{w}_n)$:
$$\overline{w}_0 = e, \quad \overline{w}_n = \overline{w}_{n-1}x_n \text{ for all } n \in \mathbb{Z}$$
Observe that $\overline{w}_{-(n+1)} = \overline{w}_{-n}x_{-n}^{-1}$ so:
$$\forall n > 0, \overline{w}_n = x_1 \dots x_n \text{ and } \overline{w}_{-n} = x_0^{-1} \dots x_{n-1}^{-1}$$
Define $p_+, p_- : \overline{\Omega} \to \Omega$ by $p_+(\overline{\mathbf{w}}) = \mathbf{w}, p_-(\overline{\mathbf{w}}) = \check{\mathbf{w}}$ for all $\overline{\mathbf{w}} = (\overline{w}_n)_{n\in\mathbb{Z}}$ where $\mathbf{w} = (w_n)_{n\geq 0}$ and $\check{\mathbf{w}} = (\check{w}_n)_{n\geq 0}$. For $n \geq 0$, $w_n = \overline{w}_n, \check{w}_n = \overline{w}_{-n}$.
- Define $\overline{P} = q_*\mu^{\otimes\mathbb{Z}}$, $\check{P} = (p_-)_*\overline{P}$. Observe that $(p_+)_*\overline{P} = P$. This follows from the fact that the following diagram commutes:
$$\begin{array}{ccc} \overline{\Omega}_{\text{step}} & \xrightarrow{q} & \overline{\Omega} \\ \big\uparrow & & \big\downarrow {\scriptstyle p_+} \\ \Omega_{\text{step}} & \xrightarrow[p]{} & \Omega \end{array}$$
where the vertical inclusion on the left maps $\mathbf{w}$ to $(\mathbf{c}, \mathbf{w})$ for some fixed $\mathbf{c} \in G^{\mathbb{Z}_{\leq 0}}$. Also observe that $\check{P}$ is the law of a random walk on G with step law given by the **reflected measure** $\check{\mu}$: $\forall g \in G, \check{\mu}(g) = \mu(g^{-1})$ starting at identity.
- Clearly $p_+ \circ q$ and $p_- \circ q$ are independent random variables and the map $(p_+, p_-) : (\overline{\Omega}, \overline{P}) \to (\Omega \times \Omega, P \otimes \check{P})$ is an isomorphism of measure spaces.

Remark. For any bilateral path $\overline{w}$ passing through the identity at time 0, $p_+(\overline{w}), p_-(\overline{w})$ give the future and past halves of the path. All that one is saying is that the future unilateral path has the same law as the random walk with step law μ while the past unilateral path has the law of the random walk with step law $\check{\mu}$. The full bilateral path is the independent coupling of past with future.

- In fact q is also an isomorphism with $\tilde{q}((\overline{w}_n)) = (\overline{w}_n\overline{w}_{n-1}^{-1})$ being an inverse, so the Bernoulli shift $\overline{U}_{\text{step}}$ on $\overline{\Omega}_{\text{step}}$ induces one on the bilateral path space: $\overline{U} = q \circ \overline{U}_{\text{step}} \circ \tilde{q}$.
- How does $\overline{U}$ act on a bilateral path $\overline{\mathbf{w}} = (\overline{w}_n)$ passing through e at time 0? Does a standard Bernoulli shift: $(\overline{w}_n) \mapsto (\overline{w}_{n+1})$ and then a change of coordinate so as to make the path pass through identity at time 0, i.e. $(\overline{w}_{n+1}) \mapsto (w_1^{-1}\overline{w}_{n+1})$.
- Note that $\overline{U}$ is an ergodic measure-preserving transformation. We immediately deduce that:

Proposition 9.1. *Let $(B_+, \lambda_+), (B_-, \lambda_-)$ be μ-,$\check{\mu}$-boundaries of (G, μ) respectively. The diagonal action of G on $(B_- \times B_+, \lambda_- \otimes \lambda_+)$ is ergodic.*

Proof. Let π denote the map $\mathrm{bnd}_- \times \mathrm{bnd}_+ \circ p_- \times p_+ : \overline{\Omega} \to B_- \times B_+$. Suppose A is a G-invariant subset of $B_- \times B_+$. For any $\overline{\mathbf{w}} \in \overline{\Omega}$,

$$\begin{aligned}\pi(\overline{U}(\overline{\mathbf{w}})) &= (\mathrm{bnd}_-(p_-(\overline{U}(\overline{\mathbf{w}}))), \mathrm{bnd}_+(p_+(\overline{U}(\overline{\mathbf{w}})))) \\ &= (\mathrm{bnd}_-(w_1^{-1}T(\mathbf{w})), \mathrm{bnd}_+(w_1^{-1}T(\check{\mathbf{w}}))) \\ &= w_1^{-1}(\mathrm{bnd}_-(\mathbf{w}), \mathrm{bnd}_+(\check{(}\mathbf{w}))) \quad (\mathrm{bnd}_\pm \text{ is } G\text{-equivariant and } T\text{-invariant}) \\ &= w_1^{-1}\pi(\overline{\mathbf{w}})\end{aligned}$$

i.e. $\pi^{-1}(A)$ is $\overline{U}$-invariant. So $\overline{P}(\pi^{-1}(A)) = 0$ or 1 i.e. $\lambda_- \otimes \lambda_+(A) = 0$ or 1. Thus the diagonal action of G on $(B_- \times B_+, \lambda_- \otimes \lambda_+)$ is ergodic. ■

We are ready to see an abstract strip approximation:

Theorem 9.1. *Let $(B_+, \lambda_+, bnd_+), (B_-, \lambda_-, bnd_-)$ be μ,$\check{\mu}$-boundaries of (G, μ) respectively. Let $\mathcal{G} = (G_k)_{k\geq 1}$ be a gauge on G with gauge function $|\cdot|$ and let $S : B_- \times B_+ \to \mathcal{P}(G)$ be a G-equivariant map, which assigns to pairs $(\gamma_-, \gamma_+) \in B_- \times B_+$, non-empty subsets $S(\gamma_-, \gamma_+)$ (“**strips**”). If:*

1. *$H(\mu)$ is finite.*
2. *$\forall g \in G$, for $\lambda_- \otimes \lambda_+$-a.e. $(\gamma_-, \gamma_+) \in B_- \times B_+$:*

$$\frac{1}{n} \log |S(\gamma_-, \gamma_+)g \cap G_{|W_n|}| \underset{n\to\infty}{\to} 0 \quad \textit{in probability.}$$

Then the boundary (B_+, λ_+) is maximal.

Proof. Let us denote the composition $\overline{\Omega} \overset{p_\pm}{\to} \Omega \overset{\mathrm{bnd}_\pm}{\to} B_\pm$ by $\Pi_\pm$. Next, observe that since all strips are necessarily non-empty, there exists some $g \in G$ for which

$$\lambda_- \otimes \lambda_+(\{(\gamma_-, \gamma_+) \in B_- \times B_+ \mid g \in S(\gamma_-, \gamma_+)\}) > 0$$

Thus if necessary, we may replace S by its g^{-1}-right translate so that:

$$\lambda_- \otimes \lambda_+(\{(\gamma_-, \gamma_+) \in B_- \times B_+ \mid g \in S(\gamma_-, \gamma_+)\}) = \bar{P}[e \in S(\Pi_-, \Pi_+)] = p > 0$$

This is useful because:

$$\begin{aligned}\bar{P}[W_n \in S(\Pi_-, \Pi_+)] &= \bar{P}[e \in S(W_n^{-1}\Pi_-, W_n^{-1}\Pi_+)] \quad (G\text{-equivariance of strips}) \\ &= \bar{P}[e \in S(\Pi_- \circ \overline{U}^n, \Pi_+ \circ \overline{U}^n)] \\ &= \bar{P}[e \in S(\Pi_-, \Pi_+)] = p \quad (\overline{U} \text{ preserves } \bar{P})\end{aligned}$$

Claim: $$\boxed{\bar{P}[W_n \in S(\Pi_-, \Pi_+)] = \iint P_n^{\gamma_+}(S(\gamma_-, \gamma_+))d\lambda_-(\gamma_-)d\lambda_+(\gamma_+)}$$

This is a simple computation.

$$\begin{aligned}\bar{P}[W_n \in S(\Pi_-,\Pi_+)] &= \check{P}\otimes P[W_n \in S(\mathrm{bnd}_-,\mathrm{bnd}_+)]\\ &= \int\left(\int P[w_n \in S(\mathrm{bnd}_-(\check{\mathbf{w}}),\mathrm{bnd}_+(\mathbf{w}))]dP(\mathbf{w})\right)d\check{P}(\check{\mathbf{w}})\\ &= \int\left(\int P^{\gamma_+}[w_n \in S(\mathrm{bnd}_-(\check{\mathbf{w}}),\gamma_+)]d\lambda_+(\gamma_+)\right)d\check{P}(\check{\mathbf{w}})\\ &= \int\left(\int P_n^{\gamma_+}(S(\mathrm{bnd}_-(\check{\mathbf{w}}),\gamma_+))d\check{P}(\check{\mathbf{w}})\right)d\lambda_+(\gamma_+)\\ &= \iint P_n^{\gamma_+}(S(\gamma_-,\gamma_+))d\lambda_-(\gamma_-)d\lambda_+(\gamma_+)\end{aligned}$$

Let $K_n = \min\{k \geq 1 \mid \mu^n(G_k) \geq 1-p/2\}$. So in words, K_n is the smallest positive integer k such that the probability of the random walk being in the k^{th} gauge set (read level set) is at least $1-p/2$.

Claim: $\boxed{\lambda_-\otimes\lambda_+\left((\gamma_-,\gamma_+)\in B_-\times B_+ \mid P_n^{\gamma_+}[S(\gamma_-,\gamma_+)\cap G_{K_n}]\geq p/4]\right)\geq p/4}$

Note that $P_n^{\gamma_+}[S(\gamma_-,\gamma_+)] \leq P_n^{\gamma_+}[S(\gamma_-,\gamma_+)\cap G_{K_n}] + (1-P_n^{\gamma_+}[G_{K_n}])$ so

$$\iint P_n^{\gamma_+}[S(\gamma_-,\gamma_+)\cap G_{K_n}]d\lambda_-(\gamma_-)\lambda_+(\gamma_+)$$

$$\geq \iint P_n^{\gamma_+}[S(\gamma_-,\gamma_+)]d\lambda_-(\gamma_-)\lambda_+(\gamma_+) + \int P_n^{\gamma_+}[G_{K_n}]\lambda_+(\gamma_+) - 1 \geq p + (1-p/2) - 1 = p/2$$

The claim now follows.
On the other hand, condition (2) says that:

$$\text{for } \lambda_-\otimes\lambda_+\text{-a.e. } (\gamma_-,\gamma_+)\forall\varepsilon>0, P\left[\frac{1}{n}\log|S(\gamma_-,\gamma_+)\cap G_{|W_n|}|\geq\varepsilon\right]\to 0$$

So since $P[|W_n| = K_n] \geq 1-p/2 > 0$, we have:

$$\text{for } \lambda_-\otimes\lambda_+\text{-a.e. } (\gamma_-,\gamma_+)\forall\varepsilon>0, P\left[\frac{1}{n}\log|S(\gamma_-,\gamma_+)\cap G_{|W_n|}|<\varepsilon \text{ and } |W_n| = K_n\right]\to 1-\frac{p}{2}>0$$

$$\text{Thus for } \lambda_-\otimes\lambda_+\text{-a.e. } (\gamma_-,\gamma_+) \quad \boxed{\lim_{n\to\infty}\frac{1}{n}\log|S(\gamma_-,\gamma_+)\cap G_{K_n}| = 0}$$

By Egoroff's theorem, we deduce that there exists a measurable set $Z \subset B_-\times B_+$ such that $\lambda_-\otimes\lambda_+(Z)\geq 1-p/8$ and $\frac{1}{n}\log|S(\gamma_-,\gamma_+)\cap G_{K_n}|$ converges to 0 *uniformly* on Z. Now let:

$$\begin{aligned}A_n(\gamma_-,\gamma_+) &= S(\gamma_-,\gamma_+)\cap G_{K_n}\\ Z_n &= Z\cap\{(\gamma_-,\gamma_+)\mid P_n^{\gamma_+}(A_n(\gamma_-,\gamma_+))\geq p/4\}\\ W &= p_+\left(\cap_{n\geq 1}\cup_{m\geq n} Z_m\right)\end{aligned}$$

Note that $\lambda_-\otimes\lambda_+(Z_n)\geq p/8$. So $\lambda_+(W)\geq \limsup_{n\to\infty}\lambda_-\otimes\lambda_+(Z_n)\geq p/8>0$. For almost every $\gamma_+\in W$, there is a $\gamma_-\in B_-$ such that for infinitely many n, $P_n^{\gamma_+}(A_n(\gamma_-,\gamma_+))\geq p/4$ i.e. $\limsup_{n\to\infty}P_n^{\gamma_+}(A_n(\gamma_-,\gamma_+))\geq p/4>0$ while $\log|A_n(\gamma_-,\gamma_+)| = o(n)$. Thus it follows from corolloary 7.3.1 that (B_+,λ_+) is maximal. ∎

Remark. Both theorem 8.1 and theorem 9.1 say that if one is able to *track* the location of the random walk conditioned to travel to a given boundary point at any time using some *nominal* data upto $o(n)$ error then our boundary is maximal.

- This idea needs to be understood more carefully
- Does theorem 9.1 say in any concrete manner that the strips approximate/track bilateral paths?

It is not quite clear how one is supposed to go about verifying condition (2) in the abstract strip approximation theorem. As we see below, if we throw in some moment conditions and give our gauge more structure, condition (2) can be turned into a deterministic criterion.

Lemma 9.1. *If $\mathcal{G}$ is a temperate gauge on G with finite first moment, then $H(\mu)$ is finite.*

Proof. ∎

Theorem 9.2 (Strip approximation)**.** *Let $(B_+, \lambda_+, bnd_+), (B_-, \lambda_-, bnd_-)$ be $\mu,\check{\mu}$-boundaries of (G, μ) respectively. Let $\mathcal{G} = (G_k)_{k\geq 1}$ be a **subadditive, temperate** gauge on G with gauge function $|\cdot|$ and let $S : B_- \times B_+ \to \mathcal{P}(G)$ be a measurable G-equivariant map. If either of the following hold then $(B_-, \lambda_-), (B_+, \lambda_+)$ are maximal:*

(1) $|\cdot|$ *has finite first moment and the strips grow subexponentially:*

$$E_\mu[|\cdot|] = \sum_{g\in G} |g|\mu(g) < \infty \quad (\textit{finite moment})$$

$$\textit{for } \lambda_- \otimes \lambda_+\textit{-a.e. } (\gamma_-, \gamma_+), \lim_{k\to\infty} \frac{1}{k} \log |S(\gamma_-, \gamma_+) \cap G_k| = 0 \quad (\textit{subexp growth})$$

(2) $|\cdot|$ *has finite entropy, finite first logarithmic moment and strips grow polynomially:*

$$H(\mu) < \infty \quad (\textit{finite entropy})$$

$$E_\mu[\log|\cdot|] = \sum_{g\in G} \log |g|\mu(g) < \infty \quad (\textit{finite logarithmic moment})$$

$$\textit{for } \lambda_- \otimes \lambda_+\textit{-a.e. } (\gamma_-, \gamma_+), \sup_{k>1} \frac{1}{\log k} \log |S(\gamma_-, \gamma_+) \cap G_k| < \infty \quad (\textit{polynomial growth})$$

Proof. We start with (1). Note that since the gauge function is subadditive and has finite first moment, using the subadditive ergodic theorem for the ergodic system (Ω, P, U), we have:

$$(*) \quad \boxed{\frac{|W_n|}{n} \to l \quad P\text{-almost surely and in } L^1(P) \quad \text{In fact } l = \inf_n \frac{E_P(|W_n|)}{n}}$$

Now let $\mathbf{w} = (w_n)\Omega$ be any sample path, $(\gamma_-, \gamma_+) \in B_- \times B_+$ and $g \in G$. Then:

$$\begin{aligned} |S(\gamma_-, \gamma_+)g \cap G_{|w_n|}| &= |S(\gamma_-, \gamma_+) \cap G_{|w_n|}g^{-1}| \\ &\leq |S(\gamma_-, \gamma_+) \cap G_{|w_n|+|g^{-1}|}| \quad (\text{Subadditivity of gauge}) \end{aligned}$$

$$\implies \frac{1}{n} \log |S(\gamma_-, \gamma_+)g \cap G_{|w_n|}| \leq \left(\frac{\log |S(\gamma_-, \gamma_+) \cap G_{|w_n|+|g^{-1}|}|}{|w_n| + |g^{-1}|}\right) \left(\frac{|w_n| + |g^{-1}|}{n}\right)$$

Now it follows from subexp growth of strips and transience of the random walk that the first factor can be arbitrarily small for sufficiently large n while $(*)$ ensures that the second factor remains bounded. Thus:

$$\forall g \in G \text{ for } \lambda_- \otimes \lambda_+\text{-a.e. } (\gamma_-, \gamma_+), \limsup_{n\to\infty} \frac{1}{n} \log |S(\gamma_-, \gamma_+)g \cap G_{|W_n|} \to 0 P\text{-a.s.}$$

This proves the maximality of (B_+, λ_+) using theorem 9.1 and lemma 9.1. The same arguments can be made for the reflected measure $\check{\mu}$ too.

Moving on to (2), it is clear that a proof analogous to that of (1) can be done once we have showed:

$$\frac{\log |W_n|}{n} \to 0 \quad P\text{-a.s.}$$

$\log |W_n| = \log(\sum_{i=1}^n |X_i|)$ and $\log |X_n|/n \to 0$ a.s. since $E_\mu[\log |X_n|] = E_\mu[\log |W_1|] < \infty$. Now:

$$\sum_{i=1}^n |X_i| \leq n \prod_{i=1}^n |X_i|$$

so we also have $\log |W_n|/n \to 0$ P-a.s. ∎

9.1. **Applications of Strip Approximation to Compactification Boundaries.** Recall the section on compactification boundaries of groups, in particular section 6.6 and theorem 6.4. We introduce another condition on a group compactification, adding to **CE**,**CP** and **CS** defined in section 6.6

Definition 9.1. Let G be a countable group and $\overline{G} = G \cup \partial G$ be a compactification of G. We say that it has property **(CG)** if:

CG: There is a left-invariant metric d on G such that the corresponding gauge $|g|_d := d(e, g)$ is temperate. For all $(\gamma_-, \gamma_+) \in \partial^{(2)}G$:

- There is at least one bi-infinite d-geodesic α in G such γ_-, γ_+ are limit points of the negative and positive rays of α respectively. We will call the set of such geodesics, the **pencil** $P(\gamma_-, \gamma_+)$.
- There is a finite set $A(\gamma_-, \gamma_+)$ such that any geodesic from $P(\gamma_-, \gamma_+)$ intersects $A(\gamma_-, \gamma_+)$.

Theorem 9.3 (Strip Approximation for Compactification Boundaries)**.** *Let $\overline{G} = G \cup \partial G$ be a separable compactification of a countable group G satisfying CE,CP,CS and CG. Let $\mu \in Prob(G)$ such that:*

1. *The subgroup generated by the support of μ is non-elementary wrt this compactification.*
2. *$H(\mu) < \infty$*
3. *μ has finite logarithmic moment wrt the gauge determined my the metric in condition **CG**.*

There exists a unique μ-stationary measure ν on ∂G and $(\partial G, \nu)$ is the Poisson boundary of (G, μ).

Proof.

- Note that **CP**, **CS** and (1) enable us to use theorem 6.4 so that for the random walk on G with step law μ:
  - we have boundary convergence: As $n \to \infty$, $W_n \to W_\infty \in \partial G$ P-a.s.

- $\lambda_+ = (W_\infty)_* P$ is the unique μ-stationary measure on ∂G and λ_+ is purely non-atomic.

Similarly for the random walk on G with step law $\check{\mu}$:
- we have boundary convergence: As $n \to \infty$, $\check{W}_n \to \check{W}_\infty \in \partial G$ $\check{P}$-a.s.
- $\lambda_- = (\check{W}_\infty)_* \check{P}$ is the unique $\check{\mu}$-stationary measure on ∂G and λ_- is purely non-atomic.

i.e. $\lambda_+ = \lambda_-$.

- Since both λ_- and λ_+ are purely non-atomic, $\lambda_- \otimes \lambda_+(\Delta) = 0$ where $\Delta = \{(x,x) | x \in \partial G\}$.
- So **CG** implies that for $\lambda_- \otimes \lambda_+$-a.e. $(\gamma_-, \gamma_+) \in \partial G \times \partial G$, there exists a minimal $M(\gamma_-, \gamma_+) > 0$ such that all geodesics from $P(\gamma_-, \gamma_+)$ intersect an M-ball.
- By the G-equivariance of pencils, it follows that the map $(\gamma_-, \gamma_+) \to M(\gamma_-, \gamma_+)$ is G-invariant so it must be constant say M_0 almost everywhere, since the diagonal action of G on $(\partial G \times \partial G, \lambda_- \otimes \lambda_+)$ is ergodic (proposition 9.1).
- Now in order to implement theorem 9.2, define $\tilde{S}(\gamma_-, \gamma_+)$ to be the union of all balls B of diameter M_0 such that every geodesic of $P(\gamma_-, \gamma_+)$ intersects B. Therefore, for any geodesic $\alpha \in P(\gamma_-, \gamma_+)$, $\tilde{S}(\gamma_-, \gamma_+) \subset N_{M_0}(\alpha)$, whence the strips $\tilde{S}$ grow linearly. Thus using strip approximation i.e. theorem 9.2, we conclude that both $(\partial G, \lambda)$ is maximal.

■

10. Pivots Á la Gouëzel

Pivots were introduced and utilized by Gouëzel in order to show linear escape with exponential tail for random walks on hyperbolic spaces with finite entropy, assuming no moment conditions. They also play a *pivotal* role in the Poisson boundary identification problem, as we will see later. More importantly, pivots essentially encode what samples paths look like.
The setting is as follows: Let (X, d) be a general δ-hyperbolic metric space and G be a countable group of isometries of X. Fix any basepoint $o \in X$ and a non-elementary measure (to be defined soon) on G. If (Z_n) denotes the random walk on G with step law μ then we wish to understand the random walk $(Z_n o)$. Following Gouëzel, we will introduce the theory of pivots with the aim to establish **linear progress with exponential tail** of this random walk:

$$\boxed{\exists \kappa > 0 \text{ such that } \mathbb{P}(d(Z_n o, o) \leq \kappa n) \leq e^{-\kappa n} \text{ for } n \text{ sufficiently large}}$$

Definition 10.1. Given any element $g \in G$, we define its **translation length** as $\tau(g) = \lim_{n\to\infty} d(g^n o, o)/n$. A group element is called **loxodromic** if it has non-zero translation length. A semi-group is said to be **non-elementary** if it contains two distinct independent loxodromic elements, i.e two loxodromic elements with disjoint fixed sets in the hyperbolic boundary of X. We call a measure μ on G **non-elementary** if the semi-group generated by its support, $\text{sgr}(\mu)$, is non-elementary.

To begin with, we consider the case when G is a free group. This will be our toy example for developing pivots.

10.1. **Toy Case: Free groups.** Consider the free group on d generators $\mathbb{F}_d = \langle a_1, \ldots, a_d \rangle$ and let $S = \{a_i, a_i^{-1} \mid 1 \leq i \leq d\}$. Consider the probability measure $\mu = \mu_S * \nu$ on $\mathbb{F}_d$ where

μ_S is the uniform probability measure on S and ν is some probability measure on $\mathbb{F}_d$. We will be analyzing the random walk on $\mathbb{F}_d$ with step law μ. In particular, we will be proving the following result:

Theorem 10.1. *Suppose $d \geq 3$ and $\nu(e) = 0$. Let (g_n) be a sequence of i.i.d. random variables with law $\mu = \mu_S * \nu$. Then there exists a $\kappa > 0$ independent of ν and d such that:*

$$\boxed{\forall n \in \mathbb{N}, \mathbb{P}(|Z_n| \leq \kappa n) \leq e^{-\kappa n}}$$

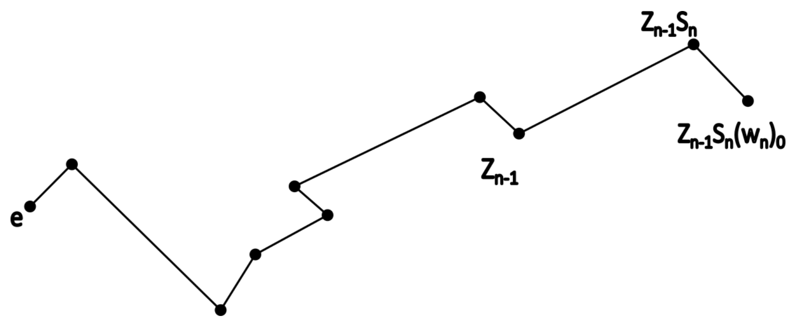
Evidently theorem 10.1 follows from the following lemma:

Lemma 10.1. *Suppose $d \geq 3$ and $\nu(e) = 0$. Fix a sequence (w_n) of elements in $\mathbb{F}_d - \{e\}$ and let (S_n) be an i.i.d. sequence of random variables with law μ_S. Consider the process defined by: $Z_0 = e, Z_n = Z_{n-1} S_n w_n$ for $n \geq 1$. Then:*

$$\forall n \in \mathbb{N}, \mathbb{P}(|Z_n| \leq \kappa n) \leq e^{-\kappa n}$$

Remark.

- Observe that μ is a non-elementary measure so the linear escape bit in Theorem 10.1 follows from the non-amenability of $\mathbb{F}_d$ and exponential growth rate of balls, by corollary 2.1.1.
- While the assumptions $d \geq 3$ and $\nu(e) = 0$ are necessary for the proof we give below, they will no longer be necessary when we develop pivots and prove linear escape for random walks on hyperbolic spaces with general step laws.



In order to prove lemma 10.1, we need to show that the walk does not backtrack too often, and that the cost of backtracking grows exponentially with the length of the beacktracking segment.

Let γ_n denote the path in the Cayley graph of $\mathbb{F}_d$ corresponding to the walk upto time n, i.e., the concatenation of the geodesics joining e to S_1, S_1 to $S_1 w_1, \ldots, S_1 w_1 \ldots S_n$ to $S_1 w_1 \ldots S_n w_n$. A time $k \in [n]$ is called **a pivotal time wrt** n if:

P1: $S_k \neq$inverse of the last letter of Z_{k-1}
$S_k \neq$ inverse of the first letter $(w_k)_0$ of w_k.

P2: The path γ_n does not backtrack to $Z_{k-1} S_k$ after time k.

P1 says the walk goes away from the origin during two steps (S_k and $(w_k)_0$) and **P2** says that after that the walk remains in the subtree based at $Z_{k-1} S_k (w_k)_0$. Let $P_0 = \phi$ and $P_n = \{\text{pivotal times wrt } n\}$ for $n \geq 1$. Thus we can estimate the progress of the walk by understanding how the (random) sets P_n evolve.

- Observe that $P_{n+1} \subset P_n \cup \{n+1\}$. This is because at time $n+1$, either $n+1$ satisfies the *local geodesic condition* **P1** so that $P_{n+1} = P_n \cup \{n+1\}$, otherwise there is backtracking and some pivotal times are destroyed i.e. $P_{n+1} \subset P_n$.

- We will now introduce a partition on S^n which will help us understand the evolution of pivotal times. Let $\bar{s} = (s_1, \ldots, s_n), \bar{s}' = (s'_1, \ldots, s'_n) \in S^n$. We say that $\bar{s}'$ is **pivoted from** $\bar{s}$ if:
  (1) they have the same pivotal times: $P_n = P'_n$ where we abuse notation a little writing $P_n(\bar{s}) = P_n$ and $P_n(\bar{s}') = P'_n$.
  (2) $\forall k \notin P_n, s_k = s'_k$.

  This is an equivalence relation on S^n and we write $\mathcal{E}_n(\bar{s})$ as the equivalence class of $\bar{s}$, i.e. sequences $\bar{s}'$ which are pivoted from $\bar{s}$.
- **What do elements of $\mathcal{E}_n(\bar{s})$ look like?** Let $|P_n| = q$. Obviously if $q = 0$, that is there are no pivots wrt time n, then $\mathcal{E}_n(\bar{s}) = \{\bar{s}\}$. So suppose $q \geq 1$. Say k is a pivotal time wrt n and let s'_k be any element of S such that $(s_1, \ldots, s'_k, \ldots, s_n)$ satisfies **P1**.

  Claim. $(s_1, \ldots, s'_k, \ldots, s_n) \in \mathcal{E}_n(\bar{s})$.

  The point here is that **P2** depends only on the part of the walk *after* time k, i.e. it only depends on $s_{k+1}, \ldots, s_n$, after all, **P2** says that the part of γ_n starting at $Z_{k-1}s_k(w_k)_0$ does not backtrack to $Z_{k-1}s_k$, i.e. the path that is the concatenation of geodesics e to w_k, w_k to $w_k s_{k+1}$, $\ldots$, $w_k s_{k+1} \ldots s_n$ to $w_k s_{k+1} \ldots s_n w_n$ does not backtrack to e. Changing s_k to s'_k *does not change the behaviour of the bit of the walk subsequent to time* k. This phenomena is what's behind the terminology. Therefore:

  Claim (Pivoting)**.** If $k_1 < \cdots < k_q$ are the pivotal times wrt n corresponding to $\bar{s}$, and $s'_{k_1}, \ldots, s'_{k_q} \in S$ satisfy the local geodesic condition at the pivotal times $k_1, \ldots, k_q$ resp., then $\bar{s}' = (s_1, \ldots, s'_{k_1}, \ldots, s'_{k_q}, \ldots, s_n)$ is pivoted from $\bar{s}$. In fact, any $\bar{s}'$ which is pivoted from $\bar{s}$ is of this form.

  There are $|S|-1$ or $|S|-2$ choices for each s'_{k_i} depending on the local situation. Thus when conditioned to $\mathcal{E}_n(\bar{s})$, $S_{k_1}, \ldots, S_{k_q}$ remain independent but no longer identically distributed. Also, $|\mathcal{E}_n(\bar{s})| \geq q^{|S|-2}$.
- Set $A_n = |P_n|$. Then $|Z_n| \geq A_n$. Thus lemma 10.1 follows once we prove that **pivots are abundant**.

Proposition 10.1 (Abundance of pivots)**.** *$A_{n+1} \geq A_n + U$ in distribution, i.e., $\mathbb{P}(A_{n+1} \geq i) \geq \mathbb{P}(A_n + U \geq i)$ for all i, where U is a random variable independent from A_n and distributed as:*

$$\mathbb{P}(U = j) = \begin{cases} \frac{d-1}{d} & ; j = 1 \\ 0 & ; j = 0 \\ \frac{2d-3}{d(2d-2)^{|j|}} & ; j < 0 \end{cases}$$

Proof. Fix $\bar{s} = (s_1, \ldots, s_n) \in S^n$ and let $q = |P_n(\bar{s})|$. We will prove the estimate by conditioning on $\mathcal{E}_n(\bar{s})$.

Case - 1: $q = 0$. So P_{n+1} is either empty or $n+1$ is a pivotal time.

- $\mathcal{E}_n(\bar{s}) = \{\bar{s}\}$. There are at least $2d-2$ choices of s'_{n+1} such that $n+1$ satisfies the local geodesic condition. So

$$\mathbb{P}(A_{n+1} \geq 0 \mid \mathcal{E}_n(\bar{s})) = 1 > \mathbb{P}(U \geq 0)$$

$$\mathbb{P}(A_{n+1} \geq 1 \mid \mathcal{E}_n(\bar{s})) \geq \frac{2d-2}{2d} = \mathbb{P}(U = 1) = \mathbb{P}(U \geq 1)$$

Case - 2: $q \geq 1$. Consider any $(\bar{s}', s'_{n+1}) \in S^{n+1}$ with $\bar{s}' \in \mathcal{E}_n(\bar{s})$.

- As observed before, after time k_q, the walk only depends on s_i, w_j for $i > k_q$ and $j \geq k_q$. So the last letter of Z'_n is the same for all $\bar{s}'$.
- Thus there are at least $2d-2$ choices for s'_{n+1} so that $(\bar{s}', s'_{n+1})$ has $n+1$ as a pivotal time wrt $n+1$. For these *good* choices, $P_n((\bar{s}', s'_{n+1})) = P_n(\bar{s}') \cup \{n+1\}$. So

$$\mathbb{P}(A_{n+1} \geq q+1 \mid \mathcal{E}_n(\bar{s})) \geq \frac{2d-2}{2d} = \mathbb{P}(U=1)$$
$$\mathbb{P}(A_{n+1} \geq q \mid \mathcal{E}_n(\bar{s})) \geq \mathbb{P}(A_{n+1} \geq q+1 \mid \mathcal{E}_n(\bar{s})) \geq \mathbb{P}(U \geq 0)$$

- Now if s'_{n+1} is *not a good choice*, then there is backtracking. Note that since k_q is pivotal wrt n, $d(Z'_n, Z'_{k_q-1}s_{k_q}) \geq 1$. So $Z'_n s'_{n+1}$ can **at most backtrack till** $Z'_{k_q-1}s_{k_q}$ (so that k_q is no longer pivotal). This requires $s'_{n+1} = (w_{k_q})_0^{-1}$. On the other hand if $Z'_n s'_{n+1}$ does not manage to backtrack till $Z'_{k_q-1}s_{k_q}$, then for $Z'_n s'_{n+1} w_{n+1}$ to backtrack till (and beyond) $Z'_{k_q-1}s'_{k_q}$, s'_{k_q} has to be the inverse of the corresponding letter in w_{n+1}.Thus:

$$\mathbb{P}(A_{n+1} \leq q-1 \mid \mathcal{E}_n(\bar{s})) \leq \frac{1}{2d} + \frac{2}{2d}\frac{1}{2d-2} \leq \frac{2}{2d}$$

  In general, to cross $j > 1$ pivotal times one must make a specific choice of generator at the crossed pivotal times after the very first one, which happen with at most probability $\frac{1}{2d-2}$. So for $j \geq 1$:

$$\mathbb{P}(A_{n+1} \leq q-j \mid \mathcal{E}_n(\bar{s})) \leq \frac{2}{2d}\frac{1}{(2d-2)^{j-1}}$$

  Thus we have for $j \geq 1$:

$$\mathbb{P}(A_{n+1} \leq q-j \mid \mathcal{E}_n(\bar{s}) \leq \mathbb{P}(U \leq -j)$$
$$\mathbb{P}(A_{n+1} \geq q+1 \mid \mathcal{E}_n(\bar{s})) \leq \mathbb{P}(U \geq 1)$$

  from which we get:

$$\forall i, \mathbb{P}(A_{n+1} \geq i \mid \mathcal{E}_n(\bar{s})) \leq \mathbb{P}(A_n + U \geq i \mid \mathcal{E}_n(\bar{s}))$$

  As this is uniform over all $\bar{s} \in S^n$, our claim follows.

∎

10.2. **General case: Groups acting on hyperbolic metric spaces.** As we saw earlier, keeping track of pivotal times and showing their abundance gives us control on the distance of the random walk from the identity. In trying to generalize this notion to the case of groups acting on hyperbolic metric spaces, we have the following questions/prompts:

- Need to come up with a geometric analogue of the subtree in **P2** so as to encode the absence of backtracking.
- Need to understand how progress is made in a general hyperbolic space. This is related to reformulating **P1**. That is, what does a random sample path travelling to a point in the hyperbolic boundary of the space *look like* ? This also relates to the reason why we looked at measures of a specific form in the toy case. We will explain this shortly.
- Finally, we need to think about making sense of a univeral compass which allows us to change direction, just as we could pivot between sub-trees in the toy case.

Let X be a δ-hyperbolic metric space. We will say that points $x, y, z \in X$ are **C-aligned** if $(x, z)_y \leq C$. Recall that the **C-shadow** of x seen from o, denoted by $S_o(x; C)$, is defined to be the set of all points $y \in X$ such that o, x, y are C-aligned. Here is the idea for *making progress* in a hyperbolic metric space:

Suppose $x_0, \dots, x_n$ is a sequence of points in X such that x_{i+1} is in the C-shadow of x_i as seen from x_{i-1} for some fixed $C \geq 0$ and for all $0 < i < n$. If the distance between consecutive points is bounded from below by a sufficiently large number $D = D(C, \delta)$ then x_n is at least at a distance of n from x_0.

Given that the step law of our random walk is non-elementary, we will see that we can keep track of certain **pivotal times** in a random sample path such that the positions of the walk at these times look just like those described in the box above. Before we move on to make all this precise, we will formalize such type of sequences, record some of their properties and also introduce a coarser version of a shadow, more suitable for our purposes.

Remark. Note that in what follows it may help to read x, y, z *are C-aligned* as z **lies in the C-shadow of y seen from** x for the sake of geometric intuition.

Lemma 10.2. *Let $x, y, z \in X$ be C-aligned. Then:*

$$d(x, z) \geq d(x, y) - C, \quad d(x, z) \geq d(y, z) - C$$

Lemma 10.3. *Suppose $w, x, y, z \in X$ are such that: w, x, y are C-aligned, x, y, z are $(C+\delta)$-aligned, and $d(x, y) \geq 2(C + \delta) + 1$. Then w, x, z are $(C + \delta)$-aligned.*

Definition 10.2. For $C, D \geq 0$, a sequence of points $x_0, \dots, x_n$ will be called a **(C, D)-chain** if x_{i-1}, x_i, x_{i+1} are C-aligned (i.e. x_{i+1} lies in the C-shadow of y seen from x) for all $0 < i < n$ and $d(x_i, x_{i+1}) \geq D$ for all $0 \leq i < n$.

This is the type of sequence we mentioned in the box.

Lemma 10.4. *Let $x_0, \dots, x_n$ be a (C, D)-chain with $D \geq 2(C + \delta) + 1$. Then*
- *(Direction) x_n lies in the $(C + \delta)$-shadow of x_1 as seen from x_0.*
- *(Progress)*

$$d(x_0, x_n) \geq \sum_{i=0}^{n-1} (d(x_i, x_{i+1}) - 2(C + \delta)) \geq n$$

Lemma 10.5. *Let $x_0, \dots, x_n$ be a (C, D)-chain with $D \geq 2(C + 2\delta) + 1$. Then for all $0 < i < n$, x_0, x_i, x_n are $(C + 2\delta)$-aligned.*

Suppose $x, y, z \in X$ and some $C \geq 0$. If $z \in S_x(y; C)$ then do we have $S_y(z; C) \subseteq S_x(y; C)$? Not necessarily.

Definition 10.3. Let $C \geq 0$ and $y, y^+, z \in X$. z is said to lie in the **C-chain-shadow** of y^+ seen from y, if there is a $(C, 2(C + \delta) + 1)$-chain $y = x_0, x_1, \dots, x_n = z$ such that y, y^+, x_1 are C-aligned. $\mathcal{C}S_y(y^+; C)$ will denote the set of all points $z \in X$ which lie in the C-chain-shadow of y^+ seen from y.

Note that if $z \in \mathcal{C}S_x(y; C)$ then $\mathcal{C}S_y(z; C) \subseteq \mathcal{C}S_x(y; C)$.

Lemma 10.6 (Comparing shadows and chain-shadows)**.** *Let $y, y^+ \in X$ and $C \geq 0$. We have:*

$$S_y(y^+; C) \subset \mathcal{C}S_y(y^+; C) \subset S_y(y^+; 2C + \delta)$$

A universal means of changing direction will be enabled by the notion of Schottky set.

Definition 10.4. Let o be a fixed basepoint in X and $\eta, C, D \geq 0$. We call a finite set of isometries S of X an (η, C, D)-**Schottky set** if:

(1) $\forall x, y \in X$

$$\frac{|\{s \in S \mid x, o, sy \text{ are } C\text{-aligned}\}|}{|S|} \geq 1-\eta, \frac{|\{s \in S \mid x, o, s^{-1}y \text{ are } C\text{-aligned}\}|}{|S|} \geq 1-\eta$$

(2) $\forall s \in S \; d(o, so) \geq D$

Proposition 10.2 ([Gou22], Corollary 3.13)**.** *Let G be a countably infinite group of isometries of X and let μ be a non-elemenatary probability measure on G. For all $\eta > 0$, there exist a $C > 0$ such that for all $D > 0$, there is a positive integer N and an (η, C, D)-Schottky set in the support of μ^N.*

We fix $\eta = 1/100$ and a $C_0 > 0$ provided by proposition 10.2. Fix $D >> C_0$. Then by proposition 10.2 there is a positive integer N and a $(1/100, C_0, D)$-Schottky set S contained in the support of μ. Let $\alpha = \min\{\mu(s) \mid s \in S^2\}$. Then there is a probability measure ν on G such that $\mu^{2N} = \alpha\mu_S^2 + (1-\alpha)\nu$, where μ_S is the uniform measure on the set S. We can re-write our random walk so as to reflect this decomposition.

- Let $(A_i)_{i\geq1}, (B_i)_{i\geq1}$ be sequences of S-valued i.i.d. random variables with law μ_S. Define $S_i = A_iB_i$ for all $i \geq 1$.
- Let $(H_i)_{i\geq1}$ be a sequence of G-valued random variables with law ν.
- Let $(\varepsilon_i)_{i\geq1}$ be a sequence of i.i.d. random variables with law given by: $\mathbb{P}(\varepsilon_i = 1) = \alpha, \mathbb{P}(\varepsilon_i = 0) = 1-\alpha$
- Consider the random variables $(\gamma_i)_{i\geq1}$ defined by:

$$\gamma_i = \begin{cases} S_i & ; \varepsilon_i = 1 \\ H_i & ; \varepsilon_i = 0 \end{cases}$$

  Clearly, (γ_i) are i.i.d. random variables with law μ. Thus at each step of our random walk, we perform a (biased) coin toss to decide if we want to change direction or walk straight ahead. In order to understand the random walk from this point of view, we study the following stochastic process.

Let $(w_i)_{i\geq0}$ be a sequence of isometries of X.Our goal now is to study the process given by $(w_0S_1w_1 \dots S_nw_no)_{n\geq1}$. Define:

$$\begin{aligned} y_i^- &= w_0S_1w_1 \dots w_{i-1}S_{i-1}o \\ y_i &= w_0S_1w_1 \dots w_{i-1}S_{i-1}A_io \\ y_i^+ &= w_0S_1w_1 \dots w_{i-1}S_{i-1}A_iB_io \end{aligned}$$

Observe that $d(y_i^-, y_i), d(y_i, y_i^+) \geq D$ since A_i, B_i are elements of the Schottky set. But what about the distance between y_i^+ and y_{i+1}^-? We will remember those *times* when we have good control on this quantity.

We will inductively define a sequence of random subsets $P_n \subset [n]$. Elements of P_n will be called **pivotal times wrt** n.

- Set $P_0 = \phi$.

- For $n \geq 1$, suppose P_{n-1} has already been defined. Then, either n is a pivotal time and $P_n = P_{n-1} \cup \{n\}$ or some pivots are destroyed and we set $P_n = P_{n-1} \cap [m]$.
- Let $k = k(n)$ be the last pivotal time before n, i.e. $k(n) = \max(P_{n-1})$. Define $k(1) = 0$ and $y_0 = o$.
- We define $P_n = P_{n-1} \cup \{n\}$ if the **local geodesic condition is satisfied at time** n, i.e.:
  P: $y_k, y_n^-, y_n, y_n^+, y_{n+1}^-$ are C_0-aligned:
  - $*$ *Rear.* y_k, y_n^-, y_n are C_0-aligned.
  - $*$ *Middle.* y_n^-, y_n, y_n^+ are C_0-aligned.
  - $*$ *Front.* y_n, y_n^+, y_{n+1}^- are C_0-aligned.

  otherwise *there is backtracking.* Let $m \in P_{n-1}$ be the largest time for which $y_{n+1}^- \in \mathcal{C}S_{y_m}(y_m^+; C_0 + \delta)$ and set $P_n = P_{n-1} \cap [m]$. If there is no such m, set $P_n = \phi$.

Before studying how the random sets (P_n) evolve, we record some important observations:

Lemma 10.7 (Pivot-shadow lemma)**.** *Suppose $P_n \neq \phi$. Let $m = \max(P_n)$. Then $y_{n+1}^- \in \mathcal{C}S_{y_m}(y_m^+; C_0 + \delta)$.*

Lemma 10.8. *Let $P_n = \{k_1 < \cdots < k_q\}$. Then $y_{k_1}^-, y_{k_1}, y_{k_2}^-, y_{k_2}, \ldots, y_{k_q}^-, y_{k_q}, y_{n+1}^-$ is a $(2C_0 + 3\delta, D - (2C_0 + 3\delta))$-chain.*

Lemma 10.9. *Let $P_n = \{k_1 < \cdots < k_q\}$. Then $o, y_{k_1}, y_{k_2}^-, y_{k_2}, \ldots, y_{k_q}^-, y_{k_q}, y_{n+1}^-$ is a $(2C_0 + 4\delta, D - (2C_0 + 3\delta))$-chain.*

Proposition 10.3. $d(o, y_{n+1}^-) \geq |P_n|$

We will now introduce a partition on $(S^2)^n$ which will help us understand the evolution of pivotal times. Let $\bar{s} = (s_1 = a_1 b_1, \ldots, s_n = a_n b_n), \bar{s}' = (s_1' = a_1' b_1', \ldots, s_n' = a_n' b_n') \in (S^2)^n$. We say that $\bar{s}'$ **is pivoted from** $\bar{s}$ if:

1. (1) they have the same pivotal times: $P_n = P_n'$ where we abuse notation a little, writing $P_n(\bar{s}) = P_n$ and $P_n(\bar{s}') = P_n'$.
2. (2) $\forall k \leq n, b_k = b_k'$.
3. (3) $\forall k \notin P_n, a_k = a_k'$.

This is an equivalence relation on $(S^2)^n$ we denote the equivalence class containing $\bar{s}$ by $\mathcal{E}_n(\bar{s})$: these are the sequences $\bar{s}' \in (S^2)^n$ which are pivoted from $\bar{s}$. **What do elements of $\mathcal{E}_n(\bar{s})$ look like?** Let $q = |P_n|$. Of course if $q = 0$, then $\mathcal{E}_n(\bar{s}) = \{\bar{s}\}$. For $q \geq 1$, we have the following observation:

Lemma 10.10. *Suppose i is a pivotal time of $\bar{s}$. Let $a_i' \in S$ be any element such that the local geodesic condition still holds at time i for the modified sequence $\bar{s}' = (s_1, \ldots, s_i' = a_i' b_i, \ldots, s_n)$. Then $\bar{s}' \in \mathcal{E}_n(\bar{s})$.*

Proof. Since conditions (2), (3) are already satisfied, we only need to show that $\bar{s}$ and $\bar{s}'$ have the same pivotal times. Note that P_j only depends on $w_0, S_1, W_1, \ldots, S_j, W_j$. So $P_{i-1} = P_{i-1}'$. Since the local geodesic condition holds at time i for both $\bar{s}$ and $\bar{s}'$, we have $P_i = P_i'$. For $j > i$, P_j only depends on $B_i, w_i, \ldots, A_j, B_j, w_j$ (to see this, inspect the local geodesic condition (**P**) and recall that d is G-invariant), which are the same for $\bar{s}$ and $\bar{s}'$. So $P_n = P_n'$. ∎

Proposition 10.4 (Abundance of pivots)**.** *Let $A_n = |P_n|$ be the number of pivotal times wrt n. Then, $A_{n+1} \geq A_n + U$ in distribution, i.e. $\mathbb{P}(A_{n+1} \geq i) \geq \mathbb{P}(A_n + U \geq i)$ for all i, where*

U is a random variable independent from A_n and distributed as:

$$\mathbb{P}(U=j)=\begin{cases}\frac{9}{10} & ; j=1\\ 0 & ; j=0\\ 9\left(\frac{1}{10}\right)^{-j+1} & ; j<0\end{cases}$$

Proof. Fix $\bar{s}=(s_1=a_1b_1,\dots,s_n=a_nb_n)\in(S^2)^n$ and let $q=|P_n(\bar{s})|$. We will prove the inqeuality by conditioning on $\mathcal{E}_n(\bar{s})$.

Claim (A)**.** $\mathbb{P}(A_{n+1}=q+1\mid\mathcal{E}_n(\bar{s}))\geq\frac{9}{10}$

Proof of claim A. $A_{n+1}=A_n+1$ if and only if $n+1$ is pivot, i.e. the local geodesic condition is satisfied at time $n+1$. Let $k=\max(P_n)$. Observe that for $k<j\leq n$, A_j takes the same values over all elements of $\mathcal{E}_n(\bar{s})$. Observe that:

- *Front* $(y_{n+1},y^-_{n+2})_{y^+_{n+1}}=(b^{-1}_{n+1}o,w_{n+2}o)_o$
- *Middle* $(y^-_{n+1},y^+_{n+1})_{y_{n+1}}=(a^{-1}_{n+1}o,b_{n+1}o)_o$
- *Rear* $(y_k,y_{n+1})_{y^-_{n+1}}=((b_kw_k\dots b_nw_n)^{-1}o,a_{n+1}o)_o$

By Schottky property, probability of choosing b_{n+1} such that *front* holds, is at least 99/100. Having chosen b_{n+1}, again by Schottky property, probability of choosing a_{n+1} such that *middle* and *rear* hold is at least 98/100. Thus, $\mathbb{P}(A_{n+1}=q+1\mid\mathcal{E}_n(\bar{s}))\geq\frac{99}{100}\cdot\frac{98}{100}\geq\frac{9}{10}$. ∎

Claim (B)**.** For all $j\geq 0$, $\mathbb{P}(A_{n+1}<q-j\mid\mathcal{E}_n(\bar{s}))\leq\left(\frac{1}{10}\right)^{j+1}$.

Proof of claim B. To begin with, claim (A) can be re-written as $\mathbb{P}(n+1 \text{ is pivotal}\mid\mathcal{E}_n(\bar{s}))\geq\frac{9}{10}$. Now: $\mathbb{P}(A_{n+1}<q-j\mid\mathcal{E}_n(\bar{s}))=\mathbb{P}(A_{n+1}<q-j\mid\mathcal{E}_n(\bar{s}),n+1 \text{ is not pivotal})\times\mathbb{P}(n+1 \text{ is not pivotal}\mid\mathcal{E}_n(\bar{s}))$. We will prove claim (B) for $j=1$ and the general claim will follow by induction.

- Pick any $\bar{s}'\in\mathcal{E}_n(\bar{s})$ with s'_{n+1} such that $n+1$ is not pivotal and there is backtracking. Let $m<k$ be the last two pivotal times wrt n, i.e. $k=\max(P_n)$ and $m=\max(P_n-\{k\})$.
- **When is there no backtracking beyond time k?**
  In other words, when does m continue to remain pivotal? Precisely when $y^-_{n+1}\in\mathcal{C}S_{y_m}(y^+_m;C_0+\delta)$. Now we know that m is the largest pivotal time before time k, for all sequences pivoted from $\bar{s}$. Thus using the pivot-shadow lemma 10.7 for time $k-1$, we have $\underline{y^-_k\in\mathcal{C}S_{y_m}(y^+_m;C_0+\delta)}$. This means that we have a chain $y_m=x_0,x_1,\dots,x_i=y^-_k$ with y_m,y^+_m,x_1 $(C_0+\delta)$-aligned. It turns out that there are plenty of choices for a'_k so that the sequence $y_m=x_0,x_1,\dots,x_i=y^-_k,\underline{x_{i+1}=y^-_{n+1}}$ is an appropriate chain, so that $y^-_{n+1}\in\mathcal{C}S_{y_m}(y^+_m;C_0+\delta)$.

Claim (C)**.** Let $\bar{s}'$ be a sequence pivoted from $\bar{s}$ such that $n+1$ is not a pivot. Let $m<k$ be the last two pivotal times wrt time n. If $a'_k\in S$ such that $x_{i-1},y^-_k,y_k,y^-_{n+1}$ are C_0-aligned, then $y_m=x_0,x_1,\dots,x_i=y^-_k,\underline{x_{i+1}=y^-_{n+1}}$ is a $(C_0+\delta,2(C_0+2\delta)+1)$-chain.

Proof of claim C. Note that $y_m=x_0,x_1,\dots,x_i=y^-_k$ is a $(C_0+\delta,2(C_0+2\delta)+1)$-chain. So it is enough to show that:

- $d(y^-_k,y^-_{n+1})\geq 2(C_0+2\delta)+1$
- $(x_{i-1},y^-_{n+1})_{y^-_k}\leq C_0+\delta$

For the first point, note that $d(y_k^-, y_{n+1}^-) \geq d(y_k^-, y_k) - (y_k^-, y_{n+1}^-)_{y_k} \geq D - C_0$ and that $D >> C_0$. Next note that $x_{i-1}, y_k^-, y_k, y_{n+1}^-$ are C_0-aligned and $d(y_k^-, y_k) = d(o, a_k o) \geq D >> C_0$, so by lemma 10.3, $x_{i-1}, y_k^-, y_{n+1}^-$ are $(C_0 + \delta)$-aligned. This takes care of the second point. ■

- **For how many choices of a_k' is the sequence $x_{i-1}, y_k^-, y_k, y_{n+1}^-$ C_0-aligned?** This happens exactly when $(x_{i-1}, y_k)_{y_k^-} = ((y_k^-)^{-1} x_{i-1}, a_k o)_o \leq C_0$ and $(y_k^-, y_{n+1}^-)_{y_k} = (a_k^{-1} o, (b_k w_k \dots b_n w_n) o)_o \leq C_0$. By Schottky property, there are at least $98|S|/100$ such choices of $a'k$!
- Therefore the conditional probability that there is backtracking beyond k, given that $n+1$ is not pivotal and $\mathcal{E}_n(\bar{s})$ is at most $(1 - 98/100)/(98/100)$ which is lesser than $1/10$. $\mathbb{P}(A_{n+1} < q - 1 \mid \mathcal{E}_n(\bar{s}), n+1 \text{ is not pivotal}) \leq \mathbb{P}(k \text{ is not pivotal wrt } n+1 \mid \mathcal{E}_n(\bar{s}), n+1 \text{ is not pivotal}) \leq 1/10$. This proves the case $j = 1$.

■

Claims (A) and (B) imply that $\mathbb{P}(A_{n+1} \geq i \mid \mathcal{E}_n(\bar{s})) \geq \mathbb{P}(A_n + U \geq i \mid \mathcal{E}_n(\bar{s}))$ for all i. And this proves the proposition. ■

11. Indentifying Poisson boundaries without logarithmic moment conditions

Our primary aim in this section is to give a proof of the following theorem from [CFFT22]:

Theorem 11.1. *Let G be a non-elementary hyperbolic group and $\mu \in Prob(G)$ be a generating probability measure with finite entropy. Let ∂G be the Gromov boundary of G and ν be the hitting measure on ∂G induced by the random walk on G with step law μ. Then the Poisson boundary of (G, μ) is $(\partial G, \nu)$.*

[CFFT22]'s proof requires three main ingredients: studying *transformed* random walks (obtained by stopping the random walk at appropriate times, to be described more precisely below), Kaimanovich's entropy criterion and the *pin-down approximation method.* Since our step law does not have finite logarithmic moment apriori, the strip approximation method is not available for use. Instead, we use the pin-down approximation method, which is essentially **an information-theoretic version of sublinear tracking of random sample paths**.

Informally, here's the idea: fix a boundary point $\xi \in \partial G$ and consider the conditional random walk, conditioned to hit ξ at time infinity. Let $\mathcal{A}_n$ be the partition of the path space that segregates sample paths by their location at time n, i.e. for any two sample paths $\mathbf{x}, \mathbf{x}'$, $\mathbf{x} \overset{\mathcal{A}_n}{\sim} \mathbf{x}'$ if $x_n = x_n'$. A sequence of partitions $(\mathcal{P}_n)$ is said to **pin down** the random walk if the conditional entropy

$$\boxed{H_\xi(\mathcal{A}_n \mid \mathcal{P}_n) = o(n)}$$

The sequence $(\mathcal{P}_n)$ will be constructed using the method of pivots, which, together with the stopping time trick, shows that a sample path of a random walk on a hyperbolic group can be thought of as a concatentation of long geodesic-like segments, attached along pivots. The fact that $H_\xi(\mathcal{A}_n \mid \mathcal{P}_n)$ grows sublinearly in n, means that $\mathcal{P}_n$ models the location of the random walk at time n well enough.

11.1. The stopping time trick.

We urge the reader to read the discussion following proposition 10.2. In what follows, we will prove the following:

Proposition 11.1. *Let G be a countable set of isometries of a δ-hyperbolic space X and $\mu \in Prob(G)$ be a non-elementary probability measure on it. For all $\varepsilon > 0$, there exists a $C > 0$ such that for any $D > 0$, there exists a probability measure θ on G for which the following hold true:*

1. *There exists an (ε, C, D)-Schottky set S such that $\theta = \kappa * \mu_S^2$ where $\kappa \in Prob(G)$ and μ_S is the uniform probability measure on S.*
2. *(G, θ) and (G, μ) have the same Poisson boundaries.*
3. *If $H(\mu) < \infty$, then $H(\theta) < \infty$.*

Using proposition 10.2 there exists an (ε, C, D)-Schottky set S and $N \in \mathbb{N}_{>0}$, $\alpha > 0$, $\nu \in \mathrm{Prob}(G)$ such that $\mu^N = \alpha\mu_S^2 + (1-\alpha)\nu$. We wish to stop the random walk whenever we pick an element of S^2, record our position and start afresh, until we pick another element of S^2, whence we repeat the process.

Recall that, a **stopping time** for a random walk is a measurable function $\tau : \Omega \to \mathbb{N}$ defined on the path space of the random walk, such that for all $n \in \mathbb{N}$, $\tau^{-1}(n) \in \mathcal{F}_n = \sigma(W_0, \dots, W_n)$, the σ-algebra generated by the position of the random walk between time 0 and n. We will only be concerned with stopping times that are finite P-almost everywhere. Now, given a stopping time τ, we define the **first return measure** μ_τ on G as follows:

$$\mu_\tau(g) = \mathbb{P}[W_\tau = g] = P(\{\mathbf{w} \in \Omega \mid w_{\tau(\mathbf{w})} = g\})$$

The random walk with step law μ_τ is called the *transformation of the random walk driven by μ determined by the stopping time τ*. In fact, the random walk with step law μ_τ is obtained by restricting the source random walk to the sequence of iterations of the stopping time τ. More precisely, we define a sequence of stopping times $(\tau_n)_{n\geq 0}$ as follows:

$$\tau_0 = 0$$
$$\tau_{n+1} = \tau_n + \tau \circ U^{\tau_n}$$

where recall that U is the Bernoulli shift induced on the path space by the Bernoulli shift on the step space and $U((w_n)_{n\geq 1}) = (w_1^{-1} w_{n+1})_{n\geq 1}$. Then the random walk with step law μ_τ is given by the random sequence $(W_{\tau_n})_{n\geq 1}$.

Theorem 11.2. *Let τ be a stopping time for the random walk on G with step law μ. Then the Poisson boundary of (G, μ) is the same as that of (G, μ_τ).*

In order to prove theorem 11.2, we need a few lemmas, which are interesting in their own right.

Lemma 11.1. *Suppose N is a normal subgroup of G and $\check{\mu} = p_*\mu$ where $p : G \to G/N$ is the canonical projection homomorphism. Then $H^\infty(G/N, \check{\mu})$ is isomorphic to the space of all bounded μ-harmonic functions on G that are N-invariant.*

Proof. By definition,

$$\forall g \in G, \check{\mu}(gN) = \sum_{n\in N} \mu(gn)$$

Also, a function $f : G \to \mathbb{R}$ is N-invariant, i.e. $\forall g \in G \forall n \in N, f(gn) = g(g)$, if and only if the function $\check{f} : G/N \to \mathbb{R}$ given by $f(gN) = f(g)$ for all $g \in G$ is well-defined. Note that G is countable so we have a countable sequence of distinct elements $(a_i)_{i\geq 1}$ such that G is a disjoint union of the cosets $(a_iN)_{i\geq 1}$. Suppose f is a bounded μ-harmonic function that is N-invariant. Now observe the following:

$$\begin{aligned}\check{f}(gN) = f(g) = \sum_{h\in G}\mu(h)f(gh) &= \sum_{i=1}^{\infty}\sum_{n\in N}\mu(a_in)f(ga_in)\\ &= \sum_{i=1}^{\infty} f(ga_i)\sum_{n\in N}\mu(a_in) \quad \text{[By } N\text{-invariance]}\\ &= \sum_{i=1}^{\infty}\check{f}(ga_iN)\check{\mu}(a_iN)\end{aligned}$$

Thus $\check{f}$ is $\check{\mu}$-harmonic. The same observation also shows that given any bounded $\check{\mu}$-harmonic function $\check{f}$ on G/N, we can lift it to a unique bounded N-invariant μ-harmonic function f on G.

■

Lemma 11.2. *Let $F = F(W)$ be the free semi-group generated by a finite or countably infinite set W. Let μ be a probability measure supported on W. Then the Poisson boundary of the random walk on F with step law μ is space $F_\infty = (F^{\mathbb{N}}, \mu^{\otimes\mathbb{N}})$.*

This follows from the simple fact that the position of random walk on free semi-group at time n uniquely describes all its steps before time n. Using lemma 11.2, it follows that:

Lemma 11.3. *Let $F = F(W)$ be the free semi-group generated by a finite or countably infinite set W. Let μ be a probability measure supported on W and τ be a stopping time for the random walk on F with step law μ. Then the Poisson boundary of (F, μ) is the same as that of (F, μ_τ), namely the space of infinite words F_∞.*

Proof of theorem 11.2. We will prove the equivalent statement that the respective spaces of harmonic functions coincide.

- Let F be the free semi-group generated by $\mathrm{supp}(\mu)$ and define $\bar{\mu}(w) = \mu(w)$ for all $w \in \mathrm{supp}(\mu)$ and zero otherwise. Let $\varphi : F \to G$ be the canonical homomorphism.
- The stopping time τ induces a stopping time $\bar{\tau}$ for the random walk on F with step law $\bar{\mu}$, by $\bar{\tau}(\bar{\mathbf{w}}) = \tau(\Phi(\bar{\mathbf{w}}))$ for all sample paths $\bar{\mathbf{w}}$ for the random walk on F, where $\Phi = \varphi^{\mathbb{N}}$.
- Now by lemma 11.3, $(F, \bar{\mu})$ and $(F, \bar{\mu}_{t\bar{a}u})$ have the same Poisson boundary. But $G = F/\ker(\varphi)$, so by lemma 11.1, we're done.

■

We state a result from [For17]:

Theorem 11.3. *Let τ be a stopping time with finite expectation. Then $H(\mu_\tau)$ is also finite and we have the following relation between asymptotic entropies for the random walks driven by μ and μ_τ:*

$$h(\mu_\tau) = E(\tau)h(\mu)$$

Now we are ready to give a proof of proposition 11.1.

Proof of proposition 11.1.

- As observed before, there exists an (ε, C, D)-Schottky set S and $N \in \mathbb{N}_{>0}$, $\alpha > 0$, $\nu \in \text{Prob}(G)$ such that $\mu^N = \alpha\mu_S^2 + (1-\alpha)\nu$.
- Define $\tau : \Omega \to \mathbb{N}$ as follows:

$$\tau(\mathbf{w}) = \inf\{i > 0 \mid w_{iN-1}^{-1} w_{iN} \in S^2\}$$

Observe that:

$$\mu_{tau} = \sum_{k=0}^{\infty} \beta^k * \bar{\alpha}$$

where $\bar{\alpha} = \alpha\mu_S^2$, $\beta = (1-\alpha)\nu$ and so $\mu^N = \alpha + \beta$.
- τ is a stopping time which is finite P-almost everywhere. It follows from theorem 11.2 that (G, μ) and (G, μ_τ) have the same Poisson boundary. Observe that:

$$\sum_{k=0}^{\infty} \beta^k * \bar{\alpha} = (\alpha \sum_{k=0}^{\infty} (1-\alpha)^k \nu^k) * \mu_S^2$$

Set $\kappa = \alpha \sum_{k=0}^{\infty} (1-\alpha)^k \nu^k$. This proves (2).
- (3) follows from theorem 11.3.

■

Remark. Proposition 11.1 concretely justifies why we worked with measures of type θ in the section on pivots, and henceforth we shall call such measures: **alternating measures**.

11.2. **Entropy criterion revisited.** Before we proceed to describe the pin-down approximation method, we set up some notation for the sake of clarity and also state a stronger version of the entropy criterion seen in theorem 7.3.

- Consider a random walk on countable group G with step law μ. Recall that the **path space** of the random walk is $\Omega = G^{\mathbb{N}}$ equipped with the product σ-algebra and the probability measure $\mathbb{P}$, the law of the random walk starting at identity.
- Given a partition $\mathcal{C} = \{C_i\}$ of the path space, we will denote the **entropy** of the partition wrt $\mathbb{P}$ as:

$$H_{\mathbb{P}}(\mathcal{C}) = H(\mathcal{C}) = -\sum_i \mathbb{P}(C_i) \log \mathbb{P}(C_i)$$

A standard sequence of partitions associated to random walk is the one that segregates samples paths by their position at a given time. For all $\mathbf{w}, \mathbf{w}' \in \Omega$, we define $\mathbf{w} \overset{\mathcal{A}_n}{\sim} \mathbf{w}'$ if $w_n = w_n'$.
- Now suppose (B, π_B, λ) is a μ-boundary for (G, μ) where $\pi_B : (\Omega, \mathbb{P}) \to (B, \lambda)$ is the boundary map. Then by disintegration we know that for λ-almost every $\xi \in B$, the **conditional probability measures** $\mathbb{P}^\xi$ exists and $\mathbb{P} = \int_B \mathbb{P}^\xi d\lambda(\xi)$.
- Given a partition $\mathcal{C} = \{C_i\}$ of the path space, we will denote the **conditional entropy** of the partition given $\xi \in B$ as:

$$H_{\mathbb{P}^\xi}(\mathcal{C}) = H_\xi(\mathcal{C}) = -\sum_i \mathbb{P}^\xi(C_i) \log \mathbb{P}^\xi(C_i)$$

and we define the **conditional entropy of $\mathcal{C}$ given** B as the average:

$$H_B(\mathcal{C}) = \int_B H_\xi(\mathcal{C}) d\lambda(\xi)$$

- Given two partitions $\mathcal{A}$ and $\mathcal{B}$, we denote their common refinement by their *join* $\mathcal{A} \vee \mathcal{B} = \{A \cap B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$. For $\mathcal{A} = (A_i)$ and $\mathcal{B} = (B_i)$ two countable measurable paritions of the path space, we have the conditional probability $\mathbb{P}^\xi(A_i \mid B_j) := \mathbb{P}^\xi(A_i \cap B_j)/\mathbb{P}^\xi(B_j)$. Again, we have the conditional entropies:
$$H_\xi(\mathcal{A} \mid \mathcal{B}) = -\sum_{i,j} \mathbb{P}^\xi(A_i \cap B_j) \log \mathbb{P}^\xi(A_i \mid B_j)$$
$$H_B(\mathcal{A} \mid \mathcal{B}) = \int_B H_\xi(\mathcal{A} \mid \mathcal{B}) d\lambda(\xi)$$

Theorem 11.4 ([FT19])**.** *Let (B, λ) be a μ-boundary. If the conditional entropy given B of the steps, $H_B(\mathcal{A}_1)$ is finite, then there exists the limit:*
$$h(B, \lambda) = \lim_{n\to\infty} \frac{H_B(\mathcal{A}_n)}{n}$$
Moreover, $h(B, \lambda) = 0$ if and only if (B, λ) is the Poisson boundary of (G, μ).

[FT19] prove theorem 11.4 as a corollary of Kaimanovich-Sobieczky's entropy criterion for random walks on equivalence relations. The μ-boundary of interest for us, in the case of hyperbolic groups, will be the Gromov boundary, equipped with the hitting measure.

11.3. **Pin-down approximation.** Here's the precise idea of pin-down approximation which we briefly mentioned at the start of this section:

Lemma 11.4 (Abstract pin-down approximation)**.** *Let (B, λ) be a μ-boundary of (G, μ). Suppose there exists a sequence of partitions $(\mathcal{P}_n)$ such that:*

(1) $H_B(\mathcal{A}_n \mid \mathcal{P}_n) = o(n)$
(2) $H(\mathcal{P}_n) = o(n)$ *Then*
$$\lim_{n\to\infty} \frac{H_B(\mathcal{A}_n)}{n} = 0$$
.

Proof. By log-convexity we have $H_B(\mathcal{A}_n) \leq H_B(A \vee \mathcal{P}_n)$. Also note that $H_B(A \vee \mathcal{P}_n) \leq H_B(A \mid \mathcal{P}_n) + H_B(\mathcal{P}_n)$ whence our claim follows readily. ■

Remark. Together with the entropy criterion (theorem 11.4), the pin-down approximation readily identifies Poission boundaries.

We will now proceed to use theory of pivots to construct the pin-down sequence of partitions.

Constructing the pin-down partitions:

- Let G be a non-elementary hyperbolic group. Fix a finite generating set Σ. Let X be the Cayley graph of G wrt to Σ. Consider an alternating measure $\theta = \kappa * \mu_S^2$ on G. We will analyze the random walk on G with step law θ.
- Recall the discussion following proposition 10.2. Set o to be the point corresponding to the identity element in X there and recall how we recursively constructed the sequence $(P_n)_{n\geq 0}$, where for $n \geq 1$ each $P_n \subset \{1, \ldots, n\}$ is the set of pivotal times before n.

Definition 11.1. A time k is said to be **pivotal from infinity** if $k \in P_m$ for all $m \geq k$, i.e k is pivotal at time k and continues to be pivotal afterwards.

Lemma 11.5. *Let (w_n) be a sample path such that w_n converges to a boundary point $\xi \in \partial G$. There exists a universal constant $M = M(C, \delta)$ such that for any time k which is pivotal from infinity for (w_n), and any geodesic $\gamma = [e, \xi)$, we have $d(y_k, \gamma) \leq M$*

Proof. Using lemma 10.9, we know that for any $n \geq k$, the sequence $o, y_{k_1}, y_{k_2}^-, y_{k_2}, \ldots, y_{k_q}^-, y_{k_q}, y_{n+1}^-$ is a $(2C_0 + 4\delta, D - (2C_0 + 3\delta))$-chain, where $P_n = \{k_1 < \cdots < k_q\}$. $k \in P_n$ by hypothesis. Using lemma 10.5, we infer that:

$$(e, y_{n+1}^-)_{y_k^-} \leq (2C + 4\delta) + 2\delta = 2C + 6\delta$$
$$(e, y_{n+1}^-)_{y_k} \leq (2C + 4\delta) + 2\delta = 2C + 6\delta$$

Thus y_k^- and y_k lie a uniformly bounded neighbourhood of any geodesic segment $[e, y_{n+1}^-]$, and any geodesic ray $[e, \xi)$. In fact, we can take $M = 2C + 9\delta$. ■

Clearly, times pivotal from infinity mark elements that closely follow geodesic rays travelling to infinity. We wish to pay special attention to them.

- Let $\alpha, L > 0$ to decided later. For every sample path (w_n) we define the following:
- Let $n \geq 1$. **Chop the interval $[0, n]$ into equal-sized pieces of size α.** That is, for each integer $0 \leq k \leq \lceil n/\alpha \rceil$, define the time interval:

$$I_{k,\alpha} = (\alpha(k-1), \alpha k] \cap [0, n]$$

- For each interval $I_{k,\alpha}$ with $k > 0$, define:

$$t_k = \begin{cases} \text{time of the first pivot from infinity in } I_{k,\alpha} & ; \text{if atleast one such pivot exists} \\ -1 & ; \text{otherwise} \end{cases}$$

  When $t_k \neq -1$, define $p_k = w_{t_k}$, recording the pivotal element. Also, let $p_0 = e$.
- Define $\mathcal{T}_n^\alpha = \{t_1, \ldots, t_{n/\alpha}\}$. Henceforth we will call *this* set as **the set of pivotal times**.
- **Good and bad intervals:** For $1 \leq k < n/\alpha$, the interval $I_{k,\alpha}$ is said to be L-good if:

$$\forall j \in I_{k,\alpha}, d(w_{j-1}, w_j) \leq L$$

  Declare $I_{0,\alpha} = \{0\}$ to be L-good.
- An interval which is not L-good will be called L-bad. By construction, the last interval $I_{n/\alpha-1,\alpha}$ that is the one containing n, is L-bad.
- **Good distance**: We define the good distance to be the sum of the distances between consecutive good pivots.

$$D_n^{\alpha,L} = \sum_{\substack{I_{k,\alpha}, I_{k+1,\alpha} \\ L\text{-good}}} d(p_k, p_{k+1})$$

  where we sum over those k such that $0 \leq k < \lceil n/\alpha \rceil$ and both $I_{k,\alpha}, I_{k+1,\alpha}$ are L-good.
- Finally, for each bad interval we record the positions occupied by the sample path **during, before and after** the bad interval. Formally, if $I_{k,\alpha}$ is a bad interval, let

$$J_{k,\alpha} = (I_{k-1,\alpha}, I_{k,\alpha}, I_{k+1,\alpha}) \cap [0, n]$$

  and define $b_k = (g_i)_{i \in J_{k,\alpha}}$ the sequence of increments during the time period $J_{k,\alpha}$. So if $I_{k_1,\alpha}, \ldots, I_{k_s,\alpha}$ are the L-bad intervals up to time n, we define the (α, L)**-bad intervals**:

$$\mathcal{B}_n^{\alpha,L} = (b_{k_1}, \ldots, b_{k_s})$$

.

Thus given a **time scale** α and **length scale** L, we have the following random variables defined on the path space:

(1) Pivotal times $\mathcal{T}_n^\alpha$: Keeping track of the pivotal times from infinity.
(2) Good distances $D_n^{\alpha,L}$: Keeping track of the distance traveled in between good times.
(3) Bad intervals $\mathcal{B}_n^{\alpha,L}$: Recording all data during bad times.

We will now prove two key results. They informally mean:

- Bad intervals are somewhat rare
- Pivotal times, good distances and bad intervals pin down the location of the random walk pretty well

Proposition 11.2 (Bad intervals are somewhat rare)**.** *Consider the random walk with step law θ on a non-elementary hyperbolic group G. For any $\delta > 0$, there exists $\alpha_0 > 0$ such that for all $\alpha \geq \alpha_0$, there exists $L \geq 1$ for which:*

$$\boxed{\mathbb{P}(I_{k,\alpha} \textit{ is } L\textit{-bad}) \leq \delta \quad \textit{for all } n \geq 1. \textit{ for all } k < \lceil n/\alpha \rceil}$$

Proof. To begin with, let us understand when exactly do we have that $I_{k,\alpha}$ is L-bad. This happens with either of the following are true:

(a) There is no pivot from infinity in $I_{k,\alpha}$
(b) There is some $j \in I_{k,\alpha}$ such that $d(w_{j-1}, w_j) > L$

We handle each case now. Case (a) is a low probability event because pivots are abundant. Using proposition 10.4 we have:

$$\boxed{\forall n, j, k \geq 0, \mathbb{P}\Big(|P_{n+j}| \geq |P_n| + k\Big) \geq \mathbb{P}\left(\sum_{i=1}^{j} U_i \geq k\right)} \quad (*)$$

where (U_i) are i.i.d. random variables with $\mathbb{E}[U_1] > 0$ and $\mathbb{E}[e^{-t_0 U_1}] < \infty$ for some $t_0 > 0$. Thus there exists a $t > 0$ such that $\mathbb{E}[e^{-tU_1}] < 1$. So for any $\beta \geq 1$, we have:

$$\boxed{\mathbb{P}\left(\sum_{i=1}^{\beta} U_i \leq 0\right) = \mathbb{P}\left(e^{-t\sum_{i=1}^{\beta} U_i} \geq 1\right) \leq \mathbb{E}[e^{-t\sum_{i=1}^{\beta} U_i}] = (\mathbb{E}[e^{-tU_1}])^{\beta}} \quad (**)$$

When does $I_{k,\alpha}$ fail to have a single pivot from infinity? This happens if and only if for each pivotal time in $I_{k,\alpha}$, the random walk eventually backtracks to the pivotal element. Thus, putting $m = \alpha(k-1)$,

$$\begin{aligned}\mathbb{P}\Big(\text{there is no pivot from } \infty \text{ in } I_{k,\alpha}\Big) &\leq \mathbb{P}\Big(\exists \beta \geq \alpha \ : \ \#P_{m+\beta} \leq \#P_m\Big) \\ &\leq \sum_{\beta=\alpha}^{\infty} \mathbb{P}\Big(\#P_{m+\beta} \leq \#P_m\Big) \\ &\leq \sum_{\beta=\alpha}^{\infty} \mathbb{P}\Big(\sum_{i=1}^{\beta} U_i \leq 0\Big) \quad [\text{Using } (*)] \\ &\leq \sum_{\beta=\alpha}^{\infty} (\mathbb{E}[e^{-tU_1}])^{\beta} \quad [\text{Using } (**)] \\ &= \frac{(\mathbb{E}[e^{-tU_1}])^{\alpha}}{1-\mathbb{E}[e^{-tU_1}]} < \delta/2\end{aligned}$$

where in the last line we have chosen α large enough. Now for case (b), note that we can choose L large enough so that

$$\mathbb{P}\Big(d(e, g_1) \geq L\Big) \leq \frac{\delta}{2\alpha}$$

hence, since the (g_i) are i.i.d.,

$$\begin{aligned}\mathbb{P}\Big(d(w_{j-1}, w_j) \geq L \text{ for some } j \in I_{k,\alpha}\Big) &\leq \sum_{j=1}^{\alpha} \mathbb{P}\Big(d(w_{j-1}, w_j) \geq L\Big) \\ &\leq \sum_{j=1}^{\alpha} \mathbb{P}\Big(d(e, g_1) \geq L\Big) \\ &\leq \alpha \cdot \frac{\delta}{2\alpha} = \frac{\delta}{2}\end{aligned}$$

Combining the estimates proves the claim. ■

Proposition 11.3 (Pivotal times, good distances and bad intervals pin down the location of the random walk pretty well)**.** *Let G be a hyperbolic group and ∂G be its Gromov boundary. Then for all $\alpha \geq 1, L \geq 1$, the join of the partitions induced by $\mathcal{T}_n^{\alpha}, D_n^{\alpha,L}, \mathcal{B}_n^{\alpha,L}$ pins down the location W_n of the random walk at time n:*

$$H_{\partial G}(\mathcal{A}_n \mid \mathcal{T}_n^{\alpha} \vee D_n^{\alpha,L} \vee \mathcal{B}_n^{\alpha,L}) = o(n)$$

Proof.

- Fix a boundary point $\xi \in \partial G$. We split the set of indices tagging the intervals $I_{k,\alpha}$, $[0, \lceil n/\alpha \rceil]$ into a disjoint union of indices coming from good intervals and maximal chains of consecutive indices of bad intervals. Using the random variables $\mathcal{T}_n^{\alpha}, D_n^{\alpha,L}, \mathcal{B}_n^{\alpha,L}$, we can record the following data:
- <u>Steps made between pivots separated by bad intervals</u>: Let $k_i, \ldots, k_{i+r}$ be a maximal chain of indices tagging bad intervals, such that $I_{k_{i+r},\alpha}$ is **not** the last interval (containing n). So $t^- = t_{k_{i-1}}$ and $t^+ = t_{k_{i+r+1}}$ are pivotal times and $p_- = w_{t_-}$ and $p_+ = w_{t_+}$ are pivots. Let $W_i = g_{t^-+1} \ldots g_{t^+}$, be the product of steps taken during the said maximal chain of bad intervals (we know this due to $\mathcal{B}_n^{\alpha,L}$).

- Let p_{last} be the last pivot. All intervals after the last pivot are bad, so let $W_{\text{last}} = g_{t^-+1} \dots g_n$ be the product of steps taken during the last maximal chain of bad intervals following p_{last} (we know this due to $\mathcal{B}_n^{\alpha,L}$).
- Note that $w_n = p_{last}W_{last}$.
- Now pick a geodesic ray $\gamma = [e, \xi)$ and recall from lemma 11.5 that each pivot lies in an M-neighbourhood of γ. Since there are atmost n/α many pivots, the nearst point projection of p_{last} on γ is determined upto an error of $2Mn/\alpha$. Now p_{last} also lies in an M-neighbourhood of γ, so the maximum number of choices for p_{last} is equal to $(2Mn/\alpha) \times |B(2M)|$. Thus:

$$H_\xi(\mathcal{A}_n \mid \mathcal{T}_n^\alpha \vee D_n^{\alpha,L} \vee \mathcal{B}_n^{\alpha,L}) \leq \log((2Mn/\alpha) \times |B(2M)|) = o(n)$$

Integrating over ∂G proves the claim.

■

Proposition 11.4. *Let G be a hyperbolic group and ∂G be its hyperbolic boundary. Consider a random walk on G with step law given by an alternating measure $\theta = \kappa * \mu_S^2$. If θ has finite entropy then for any $\varepsilon > 0$, there exists $\alpha_0 > 0$ such that for all $\alpha \geq \alpha_0$ there exists $L \geq 1$ such that:*

$$\limsup_{n\to\infty} \frac{H(\mathcal{A}_n \mid \mathcal{T}_n^\alpha \vee D_n^{\alpha,L} \vee \mathcal{B}_n^{\alpha,L})}{n} \leq \varepsilon$$

Proof. For a proof, see [CFFT22] lemma 2.4 and theorem 4.6. ■

Thus, it readily follows from the abstract pin-down approximation theorem 11.4 that:

Theorem 11.5. *Let G be a hyperbolic group and ∂G be its Gromov boundary. Consider a random walk on G with step law given by the alternating measure $\theta = \kappa * \mu_S^2$. If θ has finite entropy then we have:*

$$\lim_{n\to\infty} \frac{H_{\partial G}(\mathcal{A}_n)}{n} = 0$$

As a corollary we deduce theorem 11.1:

Proof of theorem 11.1.

- For G acting on X(its Cayley graph wrt to a finite generating set Σ) and for $\epsilon = 1/100$ using proposition 11.1, there exists a $C > 0$ such that for all $D > 0$, there exists an integer $l > 0$ and a $(1/100, C, D)$-Schottky set S contained in the support of μ. we also have an alternating measure $\theta = \kappa * \mu_S^2$ on G such that (G, μ) and (G, θ) have the same Poisson boundary. The proposition also says since μ has finite entropy, so does θ.
- Thus we consider identifying the Poisson boundary of (G, θ). Since θ is non-elementary, almost every sample path converges to the Gromov boundary ∂G. Let λ be the hitting measure on ∂G.
- Proposition 11.1 also says since μ has finite entropy, so does θ. Thus by theorem 11.5 and the entropy criterion, i.e. theorem 11.4, we conclude that the Poisson boundary of (G, θ) is $(\partial G, \lambda)$.

■

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